

SUMMATION OF SERIES

COLLECTED BY

L. B. W. JOLLEY, M.A. (CANTAB.), M.I.E.E.

SECOND REVISED EDITION

DOVER PUBLICATIONS, INC.
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VISIT...

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PREFACE TO DOVER EDITION

A SECOND edition published in the United States of America provides an opportunity for including many new series with an increase of more than 50 per cent over the original number. It has also been possible to rearrange the series in a more reasonable form.

Some corrections have been received from readers, and useful suggestions have been made by them for the present arrangement. These are gratefully acknowledged, and it will be of great assistance for future editions if readers will communicate their ideas for further expansion.

In using this collection himself, the author has experienced difficulty in tracing certain series, and there does not seem any solution excepting a complete search through all the series given. For example, certain series including inverse products appear in different parts of the book, and, if a search is to be avoided, a complete rearrangement combined with excessive duplication would be necessary. It does not seem possible, as in the case of a collection of integrals, to arrange them in a completely rational manner. Any suggestion in this direction would be specially welcome.

Among the new series included are some of those developed by Glaisher in many publications, notably the *Quarterly Journal of Mathematics*. Basing his series on Bernoulli functions, Glaisher evolved a number of coefficients which apparently simplify the appearance of the series. In this present collection, only a few are given, and the original articles should be consulted if the reader wishes to investigate them further.

The author wishes to acknowledge permission by the London Scientific Computing Service to publish tables from the *Index of Mathematical Tables* by Fletcher, Miller, and Rosenhead—an

exceedingly useful book for any engaged in work on applied mathematics; and also to thank Mrs. H. M. Cooper for her excellent work in typing a difficult manuscript.

L. B. W. JOLLEY

*623 Upper Richmond Road West,
Richmond, Surrey, 1960*

PREFACE TO FIRST EDITION

FOR a long time past there has been a need for a collection of series into one small volume for easy reference together with a bibliography indicating at least one of the textbooks to which reference could be made in case of doubt as to accuracy or to the method by which the series was arrived at.

The 700-odd series in this collection (with the exception of a few which have been specially prepared) are not new, and represent only the labour of extracting the material from the many textbooks on algebra, trigonometry, calculus and the like. Yet such a collection will, it is felt, be of considerable benefit to those engaged in the solution of technical problems, and will save a great deal of time in searching for the required result.

Criticism may be offered on the grounds that the inclusion of easy algebraical summations is unnecessary, but they have been inserted for a very definite purpose. For example, a series of inverse products may have for its sum an expression which is simple to find; but on the other hand, the solution may entail a complicated expression involving the integration or differentiation of other series. For this reason the arrangement of the series has been difficult, and overlapping is unavoidable in certain instances. To overcome this difficulty, the series have been set forth in as pictorial a manner as possible, so that the form of the individual terms can be readily seen.

On this account also, the inclusion of such series as are evolved for elliptic integrals, Bessel functions and the like has been restricted, perhaps to too great an extent; but reference to standard works is usually essential in such cases, and practically only such references are included.

The final column refers to the bibliography at the beginning of the book, and here again it has been quite impossible for obvious reasons to provide for all the references.

One of the most useful works, if it is desired to pursue any one particular problem further, is the *Smithsonian Tables*.

The scope of many of the series can be greatly enlarged by differentiation or integration of some of the forms given, and in the case of an integrated series, the constant of integration must be obtained by suitable methods. Infinite products are often of value in obtaining new series by taking logarithms and by differentiating or integrating subsequently.

In many cases it has been impossible in this small volume to comment on the limits or assumptions made in any particular summation; particularly is this the case with oscillating series: and in case of doubt it is always safer to refer to a textbook, and to bear in mind that this collection is supplementary to, and not in place of, the usual mathematical books.

Special attention is drawn in cases of difficult summations to the General and Special Forms (pages 216–225).

In all cases $\log h$ denotes the logarithm to the Napierian base, in accordance with modern practice.

Finally, any additions or corrections would be welcomed for embodiment in subsequent editions.

L. B. W. JOLLEY

*Fairdene, Sheen Road,
Richmond, Surrey, 1925*

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BIBLIOGRAPHY

<i>Indicating Letter</i>	<i>Author</i>	<i>Title and Publisher</i>
A†	T. J. Bromwich	<i>Introduction to the Theory of Infinite Series</i> , London: Macmillan Co., 1926.
B	L. L. Smail	<i>Elements of the Theory of Infinite Processes</i> , New York: McGraw-Hill Book Co., 1923.
C	G. Chrystal	<i>Algebra, An Elementary Text Book for the Higher Classes of Secondary Schools</i> , New York: Dover Publications, Inc., 1961.
D	Levett and Davison	<i>Plane Trigonometry</i> , New York: Macmillan Co., 1892.
E	S. L. Loney	<i>Plane Trigonometry (Parts I and II)</i> , Cambridge: Cambridge University Press, 1900.
F	H. S. Hall and S. R. Knight	<i>Higher Algebra</i> , London: Macmillan Co., 1899.
G	E. T. Whittaker and G. Robinson	<i>Calculus of Observations</i> , Glasgow: Blackie and Son, 1937.
H	H. Lamb	<i>Infinitesimal Calculus</i> , Cambridge: Cambridge University Press, 1921.
J	L. Todhunter	<i>Integral Calculus</i> , London: Macmillan Co., 1880.
K	C. P. Steinmetz	<i>Engineering Mathematics</i> , New York: McGraw-Hill Book Co., 1911.
L	J. Edwards	<i>Differential Calculus for Beginners</i> , London: Macmillan Co., 1899.
M	J. Edwards	<i>Integral Calculus for Beginners</i> , London: Macmillan Co., 1898.
N	G. S. Carr	<i>Synopsis of Pure Mathematics</i> , London: Hodgson, 1886.

† In the text, the numbers preceding the *Reference* letter refer to the volume of the work cited; the numbers following the *Reference* letter refer to pages.

<i>Indicating Letter</i>	<i>Author</i>	<i>Title and Publisher</i>
O	E. W. Hobson	<i>A Treatise on Plane Trigonometry</i> , New York: Dover Publications, Inc., 1957.
P		<i>Encyclopaedia Britannica</i> , 11th edition.
Q	E. T. Whittaker and G. N. Watson	<i>Modern Analysis</i> , Cambridge: Cambridge University Press, 1920.
R	E. Goursat	<i>A Course in Mathematical Analysis</i> , Vol. 1, New York: Dover Publications, Inc., 1959.
T	E. P. Adams	<i>Smithsonian Mathematical Formulae</i> , Washington: Smithsonian Institute, 1922.
U	W. E. Byerly	<i>Fourier's Series</i> , New York: Dover Publications, Inc., 1959.
W	G. Boole	<i>Calculus of Finite Differences</i> , New York: Dover, 1960.
X	A. Eagle	<i>Fourier's Theorem</i> , New York: Longmans Green and Co., 1925.
Y	J. Edwards	<i>Differential Calculus</i> , London: Macmillan Co., 1938.
Z	J. Edwards	<i>Integral Calculus</i> , Vols. I and II, London: Macmillan Co., 1922.
AA	K. Knopp	<i>Theory and Applications of Infinite Series</i> , Glasgow: Blackie and Son, 1928.
AB	H. S. Carslaw	<i>Fourier's Series and Integrals</i> , New York: Dover Publications, Inc., 1950.
AC	Fletcher, Miller, and Rosenhead	<i>Index of Mathematical Tables</i> , London: Scientific Computing Service, 1946.
AD	E. Jahnke and F. Emde	<i>Tables of Functions</i> , New York: Dover Publications, Inc., 1945.
AE	J. W. L. Glaisher	<i>Quarterly Journal of Mathematics</i> , Vol. 29, 1898.
AE.I.	J. W. L. Glaisher	<i>Quarterly Journal of Mathematics</i> , Vol. 28, 1896.
AG	E. W. Hobson	<i>The Theory of Functions of a Real Variable</i> , Vol. II, New York: Dover Publications, Inc., 1957.

SUMMATION OF SERIES

Series No.

I. Arithmetical Progression

(1) $a + (a + d) + (a + 2d) + \dots n$ terms

II. Geometrical Progression

(2) $a + ar + ar^2 + \dots n$ terms

(3) $a + ar + ar^2 + \dots \infty$

(4) $1 + ax + a^2x^2 + a^3x^3 + \dots \infty$

III. Arithmetical and Geometrical Progression

(5) $a + (a + d)r + (a + 2d)r^2 + \dots n$ terms

(6) $a + (a + d)r + (a + 2d)r^2 + \dots \infty$

(7) $1 + 2x + 3x^2 + 4x^3 + \dots \infty$

(8) $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \dots n$ terms

(9) $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \dots n$ terms

(10) $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \dots \infty$

(11) $1 + 3x + 6x^2 + 10x^3 + \dots \infty$

† See footnote to Bibliography.

Reference†

$$= \frac{n}{2} \{2a + (n-1)d\} = \frac{n}{2} (a + l) \quad \text{where } l = \text{last term} \quad \text{F. 29}$$

$$= a \frac{(r^n - 1)}{r - 1} \quad \text{F. 39}$$

$$= \frac{a}{1-r} \quad \text{where } r < 1 \quad \text{F. 40}$$

$$= \frac{1}{1-ax} \quad \text{where } ax < 1 \quad \text{F. 158}$$

$$= \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{\{a + (n-1)d\}r^n}{1-r} \quad \text{F. 44}$$

$$= \frac{a}{1-r} + \frac{dr}{(1-r)^2} \quad \text{where } r < 1 \quad \text{F. 44}$$

$$= \frac{1}{(1-x)^2} \quad \text{where } x < 1 \quad \text{F. 44}$$

$$= \frac{35}{16} - \frac{12n+7}{16 \cdot 5^{n-1}} \quad \text{F. 45}$$

$$= 4 - \frac{1}{2^{n-2}} - \frac{n}{2^{n-1}} \quad \text{F. 45}$$

$$= 6 \quad \text{F. 45}$$

$$= \frac{1}{(1-x)^3} \quad \text{where } x < 1 \quad \text{F. 45}$$

Series No.

$$(12) \ x(x + y) + x^2(x^2 + y^2) + x^3(x^3 + y^3) + \dots \ n \text{ terms}$$

$$(13) \ \frac{2}{3} + \frac{3}{3^2} + \frac{2}{3^3} + \frac{3}{3^4} + \frac{2}{3^5} + \dots \ \infty$$

$$(14) \ \frac{4}{7} - \frac{5}{7^2} + \frac{4}{7^3} - \frac{5}{7^4} + \dots \ \infty$$

$$(15) \ 1^2 + \frac{3^2}{2} + \frac{5^2}{2^2} + \frac{7^2}{2^3} + \dots \ n \text{ terms}$$

$$(16) \ 1 + 3x + 5x^2 + \dots + (2n - 1)x^{n-1}$$

IV. Powers of Natural Numbers

$$(17) \ \sum_{x=1}^{x=n} x^p$$

$$(18) \ 1 + 2 + 3 + 4 + \dots \ n$$

$$(19) \ 1^2 + 2^2 + 3^2 + 4^2 + \dots \ n^2$$

$$(20) \ 1^3 + 2^3 + 3^3 + 4^3 + \dots \ n^3$$

$$(21) \ 1^4 + 2^4 + 3^4 + 4^4 + \dots \ n^4$$

Reference

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} + \frac{xy(x^ny^n - 1)}{xy - 1}$$

F. 46

$$= \frac{9}{8}$$

F. 46

$$= \frac{23}{48}$$

0,47916

F. 46

$$= 34 - (4n^2 + 12n + 17) \frac{1}{2^{n-1}}$$

$$= \frac{1 + x - (2n + 1)x^n + (2n - 1)x^{n+1}}{(1 - x)^2}$$

$$= \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{1}{2} \binom{p}{1} B_1 n^{p-1} - \frac{1}{4} \binom{p}{3} B_2 n^{p-3}$$

$$+ \frac{1}{6} \binom{p}{5} B_3 n^{p-5} - \dots \quad \text{where } \binom{p}{n} \text{ are the binomial coefficients}$$

and B_n are Bernoulli numbers, see No. (1129). The series ends with the term in n if p is even, and with the term in n^2 if p is odd.

T. 27

$$= \frac{n(n+1)}{2}$$

F. 50

$$= \frac{n(n+1)(2n+1)}{6}$$

F. 50

$$= \left\{ \frac{n(n+1)}{2} \right\}^2$$

F. 51

$$= \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)$$

F. 256

Series No.

$$(22) 1^5 + 2^5 + 3^5 + 4^5 + \dots n^5$$

$$(23) 1^6 + 2^6 + 3^6 + 4^6 + \dots n^6$$

$$(24) 1^7 + 2^7 + 3^7 + 4^7 + \dots n^7$$

$$(25) 1^2 + 3^2 + 5^2 + 7^2 + \dots n \text{ terms}$$

$$(26) 1^3 + 3^3 + 5^3 + 7^3 + \dots n \text{ terms}$$

$$(27) 1^3 + (1.5)^3 + 2^3 + (2.5)^3 + \dots \left(\frac{n+1}{2}\right)^3$$

$$(28) 2^2 + 4^2 + 6^2 + 8^2 + \dots n \text{ terms}$$

$$(29) 1^2 \cdot 2^1 + 2^2 \cdot 2^2 + 3^2 \cdot 2^3 + \dots n \text{ terms}$$

$$(30) 1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \dots n \text{ terms}$$

$$(31) (n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + \dots n \text{ terms}$$

$$(32) \sum_1^n (2n - 1)^2$$

$$(33) \sum_1^n (2n - 1)^3$$

$$(34) \sum_1^n (x^n + n)(x^n - n)$$

$$(35) \sum_1^n (x^n + y^n)(x^n - y^n)$$

$$(36) \sum_1^n \left(x^n - \frac{1}{x^n}\right) \left(y^n - \frac{1}{y^n}\right)$$

Reference

$$= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12} \quad \text{F. 337}$$

$$= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} \quad \text{F. 338}$$

$$= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{n^2}{12} \quad \text{F. 338}$$

$$= \frac{1}{3} n(4n^2 - 1) \quad \text{F. 256}$$

$$= n^2(2n^2 - 1) \quad \text{F. 256}$$

$$= \frac{1}{8} \left\{ \frac{(n+1)(n+2)}{2} \right\}^2 - \frac{1}{8}$$

$$= \frac{2n(n+1)(2n+1)}{3}$$

$$= 2n\{2n^2 - 4n + 6\} - 6$$

$$= \frac{1}{12} n(n+1)(n+2)(3n+5) \quad \text{F. 256}$$

$$= \frac{1}{4} n^2(n^2 - 1) \quad \text{F. 323}$$

$$= \frac{1}{3} n(2n-1)(2n+1)$$

$$= n^2(2n^2 - 1)$$

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} - \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} - \frac{y^2(y^{2n} - 1)}{y^2 - 1}$$

$$= \frac{\{(xy)^n - 1\}\{(xy)^{n+1} + 1\}}{(xy)^n(xy - 1)} - \frac{\left\{\left(\frac{x}{y}\right)^n - 1\right\}\left\{\left(\frac{x}{y}\right)^{n+1} + 1\right\} \frac{y}{n}}{\left(\frac{x}{y}\right)^n \left(1 - \frac{y}{x}\right)}$$

Series No.

$$(37) \sum_1^n \frac{x^{n+1} - y^{n+1}}{x - y}$$

$$(38) \sum_1^n (a_0 n^r + a_1 n^{r-1} + \dots a_r)$$

See No. (17)

$$(39) \sum_1^\infty x^n$$

$$(40) \sum_1^\infty nx^n$$

$$(41) \sum_1^n (1 + 2 + 3 \dots + n)x^{n-1}$$

V. Products of Natural Numbers

- (42) To find the sum of n terms of a series, each term of which is composed of r factors in arithmetical progression, the first factors of the several terms being in the same arithmetical progression: Write down the n th term, affix the next factor at the end, divide by the number of factors thus increased, and by the common difference, and add a constant.

$$(43) 1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \dots n \text{ terms}$$

$$(44) 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 \dots n \text{ terms}$$

$$(45) 2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + \dots n \text{ terms}$$

$$(46) 2 \cdot 2 + 4 \cdot 4 + 7 \cdot 8 + 11 \cdot 16 + 16 \cdot 32 \dots n \text{ terms}$$

$$(47) 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 \dots n \text{ terms}$$

$$(48) 1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + \dots n \text{ terms}$$

Reference

$$= \frac{x^2(1-x^n)}{(x-y)(1-x)} - \frac{y^2(1-y^n)}{(x-y)(1-y)}$$

$$= \{a_0 b_r + a_1 b_{r-1} + \dots + a_r n\}$$

$$b_r = 1^r + 2^r + 3^r + \dots + n^r$$

$$= \frac{x}{1-x} \quad \text{where } x < 1$$

$$= \frac{x}{(1-x)^2} \quad \text{where } x < 1$$

$$= \frac{1-x^n}{(1-x)^3} - \frac{n(n+3)x^n}{2(1-x)^2} + \frac{n(n+1)x^{n+1}}{2(1-x)^2}$$

F. 314

$$= \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{4 \cdot 2} + \frac{15}{8}$$

$$= n(2n^3 + 8n^2 + 7n - 2) \quad \text{F. 315}$$

$$= \frac{n(n+1)(n+2)}{3} \quad \text{F. 52}$$

$$= n(3n^2 + 6n + 1) \quad \text{F. 318}$$

$$= (n^2 - n + 4)2^n - 4 \quad \text{F. 333}$$

$$= \frac{1}{4} n(n+1)(n+2)(n+3) \quad \text{F. 322}$$

$$= \frac{1}{5} n(n+1)(n+2)(n+3)(n+4) \quad \text{F. 322}$$

Series No.

$$(49) 1 \cdot 4 \cdot 7 + 4 \cdot 7 \cdot 10 + 7 \cdot 10 \cdot 13 + \dots n \text{ terms}$$

$$(50) 1 \cdot 4 \cdot 7 + 2 \cdot 5 \cdot 8 + 3 \cdot 6 \cdot 9 + \dots n \text{ terms}$$

$$(51) 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 10 + 3 \cdot 7 \cdot 11 + \dots n \text{ terms}$$

$$(52) 6 \cdot 9 + 12 \cdot 21 + 20 \cdot 37 + 30 \cdot 57 + \dots n \text{ terms}$$

n th term is $(n+1)(n+2)(2n^2+6n+1)$

$$(53) 2 \cdot 2 + 6 \cdot 4 + 12 \cdot 8 + 20 \cdot 16 + 30 \cdot 32 + \dots n \text{ terms}$$

n th term is $n(n+1)2^n$

$$(54) 1 \cdot 3 \cdot 2^2 + 2 \cdot 4 \cdot 3^2 + 3 \cdot 5 \cdot 4^2 + \dots n \text{ terms}$$

$$(55) \sum_1^n (p-n)(q-n)$$

$$(56) \sum_1^n \frac{m(m+1) \dots (m+n-1)}{n!}$$

$$(57) \sum_1^n \frac{b(b+1)(b+2) \dots (b+n-1)}{a(a+1)(a+2) \dots (a+n-1)}$$

$$(58) \frac{1^2 \cdot x}{2 \cdot 3} + \frac{2^2 \cdot x^2}{3 \cdot 4} \dots n \text{ terms}$$

This series is integrable if $x = 4$.

$$(59) \sum_1^n \frac{n^2 4^n}{(n+1)(n+2)}$$

Reference

$$= \frac{1}{12} (3n - 2)(3n + 1)(3n + 4)(3n + 7) + \frac{56}{12} \quad \text{F. 322}$$

$$= \frac{1}{4} n(n + 1)(n + 6)(n + 7) \quad \text{F. 322}$$

$$= \frac{1}{4} n(n + 1)(n + 8)(n + 9) \quad \text{F. 322}$$

$$= \frac{2}{5} n(n + 1)(n + 2)(n + 3)(n + 4) \\ + \frac{1}{3} (n + 1)(n + 2)(n + 3) - 2 \quad \text{F. 331}$$

$$= (n^2 - n + 2)2^{n+1} - 4 \quad \text{F. 332}$$

$$= \frac{1}{10} n(n + 1)(n + 2)(n + 3)(2n + 3) \quad \text{F. 323}$$

$$= \frac{n(n + 1)}{6} \{(2n + 1) - 3(p + q)\} + npq$$

$$= \frac{(m + 1)(m + 2) \dots (m + 1 + n - 1)}{n!} - 1 \quad \text{C. 200}$$

$$= \frac{(m + n)!}{m! n!} - 1$$

$$= \frac{b(b + 1)(b + 2) \dots (b + n)}{(b + 1 - a)a(a + 1)(a + 2) \dots (a + n - 1)} - \frac{b}{b + 1 - a} \quad \text{T. 28}$$

$$= \sum_1^n \frac{n^2 x^n}{(n + 1)(n + 2)} \quad \text{W. 58}$$

$$= \frac{4^{n+1}}{3} \cdot \frac{n - 1}{n + 2} + \frac{2}{3} \quad \text{W. 58}$$

Series No.

VI. Figurate and Polygonal Numbers

(60) Figurate numbers—

1	1	1	1	1	1 ...
1	2	3	4	5	6 ...
1	3	6	10	15	21 ...
1	4	10	20	35	56 ...
1	5	15	35	70	126 ...

The sum to n terms of the r th order(61) Method of Differences $\Rightarrow$ due à Newton la pomme

One Series is	12	40	90	168	280	432 ...
1st Diff.		28	50	78	112	152 ...
2nd Diff.			22	28	34	40 ...
3rd Diff.				6	6	6 ...
4th Diff.					0	0

The n th term is

The sum

$$12 + 28(n-1) + \frac{22(n-1)(n-2)}{2!} + \frac{6(n-1)(n-2)(n-3)}{3!}$$

(62) $4 + 14 + 30 + 52 + 80 + 114 + \dots$ n terms(63) $8 + 26 + 54 + 92 + 140 + 198 + \dots$ n terms(64) $9 + 16 + 29 + 54 + \dots$ n terms(65) $4 + 13 + 35 + 94 + 262 + \dots$ n terms(66) $2 + 12 + 36 + 80 + 150 + 252 + \dots$ n terms

$$= \frac{1}{r!} n(n+1)(n+2)\dots(n+r-1)$$

F. 320

en général pour toutes les P.A.
 si l'ordre de la p.a. est par exemple de 3
 alors le même terme est une équation du
 'k-1' degré; donc du second degré

$$= 12n + \frac{28n(n-1)}{2!} + \frac{22n(n-1)(n-2)}{3!} + \frac{6n(n-1)(n-2)(n-3)}{4!}$$



$$= \frac{1}{12} n(n+1)(3n^2 + 23n + 46)$$

F. 326

$$= n(n+1)^2 \Rightarrow \text{pas bonne!}$$

F. 332

$$= \frac{1}{3} n(n+1)(5n+7)$$

F. 332

$$= 6(2^n - 1) + \frac{1}{2} n(n+5)$$

F. 333

$$= \frac{3}{2} (3^n - 1) + \frac{1}{6} n(n+1)(n+5) - n$$

F. 333

$$= \frac{1}{12} n(n+1)(n+2)(3n+1)$$

F. 332

Series No.

$$(67) 30 + 144 + 420 + 960 + 1890 + \dots n \text{ terms}$$

$$(68) 2 + 5 + 13 + 35 + \dots n \text{ terms}$$

$$(69) 2 + 7x + 25x^2 + 91x^3 + \dots n \text{ terms}$$

VII. Inverse Natural Numbers

$$(70) \sum_1^n \frac{1}{n} = C + \log n + \frac{1}{2n} - \frac{a_2}{n(n+1)} \\ - \frac{a_3}{n(n+1)(n+2)} - \dots$$

$$\text{Also } 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

$$(71) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \infty$$

$$(72) 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \dots \infty$$

$$(73) \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \dots \infty$$

$$(74) \frac{5}{1 \cdot 2 \cdot 3} + \frac{14}{4 \cdot 5 \cdot 6} + \dots \infty$$

$$(75) 2 \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \infty\right) \\ = 8 \left(\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{5 \cdot 6 \cdot 7} + \frac{1}{9 \cdot 10 \cdot 11} + \dots \infty\right)$$

Reference

$$= \frac{1}{20} n(n+1)(n+2)(n+3)(4n+21)$$

F. 332

$$= \frac{1}{2} (3^n - 1) + 2^n - 1$$

F. 272

$$= \frac{1 - 4^n x^n}{1 - 4x} + \frac{1 - 3^n x^n}{1 - 3x}$$

F. 272

where C = Euler's constant, see No. (1132),

$$a_2 = \frac{1}{12} \quad a_3 = \frac{1}{12} \quad a_4 = \frac{19}{120} \quad a_5 = \frac{9}{20}$$

$$\text{and } a_k = \frac{1}{k} \int_0^1 x(1-x)(2-x) \dots (k-1-x) dx$$

T. 27

$$= 7.48547 \quad \text{where } n = 1000$$

A. 325

$$= 14.39273 \quad \text{where } n = 10^6$$

$$= \log h 2 = \log 2 = \ln 2$$

F. 195

$$= 0.43882 = \frac{\pi}{4} - \frac{1}{2} \log h 2$$

H. 475

$$= \frac{1}{2} \log h 2$$

C. 252

$$= \sum_{n=1}^{\infty} \frac{9n-4}{(3n-2)(3n-1)(3n)} = \log h 3$$

C. 253

$$= \log h 4$$

C. 252

logarithme hyperbolique
 logarithme naturel
 logarithme népérien

} same!

$$e = e = 2,71828$$

Series No.

$$(76) \ 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots \infty$$

$$(77) \ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \infty$$

$$= 1 - 2 \left[\frac{1}{3 \cdot 5} + \frac{1}{7 \cdot 9} + \frac{1}{11 \cdot 13} + \dots \infty \right]$$

$$(78) \ 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \dots \infty$$

$$(79) \ 1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \dots \infty$$

$$(80) \ \frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \dots \infty$$

$$(81) \ 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \dots \infty$$

$$(82) \ 1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \dots \infty$$

$$(83) \ 1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} + \frac{1}{13} - \frac{1}{17} - \frac{1}{19} + \dots \infty$$

$$(84) \ 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \dots \infty$$

$$(85) \ 1 - \frac{1}{2} \cdot \frac{1}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} - \dots \infty$$

$$(86) \ 1 - \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} - \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \dots \infty$$

$$(87) \ \sum_1^n \frac{1}{2n-1}$$

Reference

$$= \frac{\pi}{2\sqrt{2}} = 1,110720735$$

C. 335

$$= \frac{\pi}{4} \quad 0,7853981635$$

E. 109

$$= \frac{\pi}{8} (1 + \sqrt{2}) \quad 0,9480594490$$

M. 132

$$= \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} + \log h 2 \right) \quad 0,8356488483$$

A. 189

$$= \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} - \log h 2 \right) \quad 0,3735501277$$

A. 189

$$= \frac{\pi}{3\sqrt{3}} \quad 0,6045997880$$

27 P. 281

$$= \frac{1}{4\sqrt{2}} [\pi + 2 \log h (\sqrt{2} + 1)] \quad 0,8669729874$$

A. 190

$$= \frac{1}{\sqrt{3}} \log h (2 + \sqrt{3}) \quad 0,7603459961$$

AB. 166

$$= \frac{\pi}{2\sqrt{3}} = 0,9068996820$$

A. 528

$$= \log h (1 + \sqrt{2})$$

Y. 90

$$= \frac{2}{\sqrt{3}} \log h \left(\frac{1 + \sqrt{3}}{\sqrt{2}} \right)$$

Y. 90

$$= \frac{1}{2} (C + \log h n) + \log h 2 + \frac{B_1}{8n^2} - \frac{(2^3 - 1)B_2}{64n^4} + \dots$$

 For C see No. (1132) and for B_n see No. (1129)

A. 325

Series No.

$$(88) \frac{x}{1+x} + \frac{2x^2}{1+x^2} + \frac{4x^4}{1+x^4} \dots \infty$$

$$(89) \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \dots n \text{ terms}$$

$$(90) \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \dots \infty$$

$$(91) \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} \dots \infty$$

$$(92) \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \dots \infty$$

$$(93) e \left[\frac{1}{x} - \frac{1}{1!} \frac{1}{x+1} + \frac{1}{2!} \frac{1}{x+2} - \frac{1}{3!} \frac{1}{x+3} + \dots \infty \right]$$

$$(94) \frac{1}{3} S_2 - \frac{1}{5} S_4 + \frac{1}{7} S_6 - \dots \infty$$

$$(95) \frac{1}{2} \cdot \left(\frac{2x}{1+x^2} \right) + \frac{1}{2 \cdot 4} \left(\frac{2x}{1+x^2} \right)^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \left(\frac{2x}{1+x^2} \right)^5 + \dots \infty$$

$$(96) \sum_1^{\infty} 2x \left\{ \frac{1}{n^2} e^A - \frac{1}{(n+1)^2} e^B \right\}$$

$$A = -\frac{x^2}{n^2} \quad B = \frac{x^2}{(n+1)^2} \quad C = -x^2$$

VIII. Exponential and Logarithmic Series

$$(97) 1 + ax + \frac{a^2 x^2}{2!} + \frac{a^3 x^3}{3!} + \dots \infty$$

$$(98) 1 + x \log a + \frac{(x \log a)^2}{2!} + \dots \infty$$

$$(99) x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \infty = x(1-x) + \frac{1}{2}x^2(1-x^2) + \dots \infty$$

Reference

$$= \frac{x}{1-x} \quad \text{where } (x^2 < 1) \quad \text{T. 118}$$

$$= \frac{1}{1-x} - \frac{1}{1-x^n} \quad \text{A. 24}$$

$$= \frac{x}{1-x} \quad \text{where } x^2 < 1, \text{ and}$$

$$= -\frac{1}{x-1} \quad \text{where } x > 1 \quad \text{T. 118}$$

$$= \frac{1}{x-1} \quad \text{where } x^2 > 1 \quad \text{T. 118}$$

$$= \frac{1+2x}{1+x+x^2} \quad \text{where } x < 1 \quad \text{Y. 54}$$

$$= \frac{1}{x} + \frac{1}{x(x+1)} + \frac{1}{x(x+1)(x+2)} + \dots \infty \quad \text{A. 102}$$

$$= \frac{\pi \log h 2}{8} \quad \text{where } \mathcal{S}_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n} \quad \text{A. 192}$$

$$= x \quad \text{where } |x| < 1 \quad \text{Q. 132}$$

$$= 2x\epsilon^c$$

AB. 167

$$= e^{ax} \quad \text{F. 188}$$

$$= a^x \quad \text{F. 188}$$

$$= \log h (1+x) \quad \text{where } x < 1 \quad \text{F. 191}$$

Series No.

$$(100) -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \infty$$

$$(101) 1 + 2^3 + \frac{3^3}{2!} + \frac{4^3}{3!} + \frac{5^3}{4!} + \dots \infty$$

~~$$(102) \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \dots \infty$$~~

$$(103) 2 \left\{ \frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \dots \infty \right\}$$

$$(104) \sum_1^{\infty} \frac{1}{n(4n^2 - 1)^2}$$

$$(105) (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \dots \infty$$

$$(106) \frac{2}{1} + \frac{3}{2!} + \frac{6}{3!} + \frac{11}{4!} + \frac{18}{5!} + \dots \infty$$

$$(107) \epsilon \left(1 + x + \frac{2}{2!}x^2 + \frac{5}{3!}x^3 + \frac{15}{4!}x^4 + \dots \infty \right)$$

$$(108) 1 + 2x + \frac{2^2 - 1}{2!} \cdot x^2 + \frac{3^2 - 1}{3!} \cdot \frac{x^3}{2} + \dots \infty$$

$$(109) 1 + \frac{2x^2}{2!} - \frac{x^3}{3!} + \frac{7x^4}{4!} - \frac{23x^5}{5!} + \frac{121x^6}{6!} - \dots \infty$$

$$(110) 2 \left\{ \frac{x^2}{2} - \frac{1}{3} \left(1 + \frac{1}{2} \right) x^3 + \frac{1}{4} \left(1 + \frac{1}{2} + \frac{1}{3} \right) x^4 - \dots \infty \right\}$$

$$(111) 2 \left\{ \frac{x^2}{2} + \frac{1}{3} \left(1 + \frac{1}{2} \right) x^3 + \frac{1}{4} \left(1 + \frac{1}{2} + \frac{1}{3} \right) x^4 + \dots \infty \right\}$$

$$(112) x^2 + \left(1 - \frac{1}{2} + \frac{1}{3} \right) \frac{x^4}{2} + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \right) \frac{x^6}{3} + \dots \infty$$

$$(113) x^2 + \left(1 - \frac{1}{3} + \frac{1}{5} \right) \frac{x^6}{3} + \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \right) \frac{x^{10}}{5} + \dots \infty$$

Reference

$$= \log h(1 - x) \quad \text{where } x < 1 \quad \text{F. 191}$$

$$= 15\epsilon \quad \text{F. 339}$$

$$= \log h 3 - \log h 2 \quad \text{F. 195}$$

$$= \epsilon^{-1} \quad \text{F. 196}$$

$$= \frac{3}{2} - 2 \log h 2 \quad \text{C. 253}$$

$$= \log h x \quad \text{where } 0 < x \leq 2$$

$$= 3(\epsilon - 1) \Rightarrow \sum_{n \neq 0} \frac{\phi(d_0, d_1, d_2)}{n!} = e^{\left(\frac{d_2}{2} + d_0\right) + (d_0 + d_2)} \quad \text{T. 124}$$

$$= \epsilon^{\epsilon x} \quad \text{F. 338}$$

$$= (1 + x)\epsilon^x \quad \text{T. 126} \quad -d_1$$

$$= \epsilon^x - \log h(1 + x) \quad \text{F. 338}$$

$$= \{\log h(1 + x)\}^2 \quad \text{where } x < 1 \quad \text{F. 191}$$

$$= [\log h(1 - x)]^2 \quad \text{A. 191}$$

$$= -\log h(1 + x) \cdot \log h(1 - x) \quad \text{A. 191}$$

$$= \frac{1}{2} (\tan^{-1} x) \log h \frac{1 + x}{1 - x} \quad \text{A. 191}$$

Series No.

$$\rightarrow (114) e^{-n} + e^{-9n} + e^{-25n} + \dots \infty$$

$$(115) \sum a_n \theta^n$$

$$(116) 1 - \frac{1}{2} \left(1 + \frac{1}{2}\right) + \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3}\right) - \dots \infty$$

$$(117) \sum_1^{\infty} \frac{1}{nx^n}$$

$$(118) \sum_1^{\infty} \frac{x^n}{n}$$

$$(119) \sum_1^{\infty} \frac{x^n}{n!}$$

$$(120) \sum_1^{\infty} \frac{x^n}{n(n+1)}$$

$$(121) x + \left(1 + \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \dots \infty$$

$$(122) \sum_1^{\infty} \frac{(n-1)x^n}{(n+2)n!}$$

$$(123) \sum_1^{\infty} \frac{(1^3 + 2^3 + 3^3 + \dots + n^3)}{n!} x^n$$

$$* \rightarrow (124) \sum_1^{\infty} \frac{1}{x^2 + n^2}$$

$$(125) \sum_1^{\infty} (-1)^n \frac{1}{x^2 + n^2}$$

$$= \frac{(2^{1/4} - 1)\Gamma(\frac{1}{4})}{2^{11/4}\pi^{3/4}}$$

$$\Gamma(\frac{1}{4}) = 3.6256 \quad \text{T. 144}$$

$$= \epsilon^A \quad \text{where } A = \epsilon^0 \text{ and } a_{n+1}$$

$$= \left(a_n + a_{n-1} + \dots \frac{a_{n-2}}{2!} + \dots \frac{a_0}{n!} \right) \frac{1}{n+1} \quad \text{Y. 111}$$

$$= \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2 \quad 0,5822405267 \quad \text{A. 520}$$

$$= \log \frac{x}{x-1} \quad \text{H. 460}$$

$$= \log \frac{1}{1-x} \quad \text{where } x < 1$$

$$= \epsilon^x - 1 \quad \text{trivial.}$$

$$= 1 - \left(\frac{1}{x} - 1 \right) \log \frac{1}{1-x} \quad \text{F. 338}$$

$$= \frac{1}{1-x} \log \frac{1}{1-x} \quad \text{where } x < 1$$

$$= \frac{\{(x^2 - 3x + 3)\epsilon^x - \frac{1}{2}x^2 - 3\}}{x^2} \quad \text{C. 236}$$

$$= \epsilon^x \left(x + \frac{7}{2}x^2 + 2x^3 + \frac{1}{4}x^4 \right) \quad \text{C. 235}$$

$$= \frac{\pi}{2x} \cdot \frac{\epsilon^{\pi x} + \epsilon^{-\pi x}}{\epsilon^{\pi x} - \epsilon^{-\pi x}} - \frac{1}{2x^2} \quad \text{T. 135}$$

$$= \frac{\pi}{x} \cdot \frac{1}{\epsilon^{\pi x} - \epsilon^{-\pi x}} - \frac{1}{2x^2} \quad \text{T. 135}$$

Series No.

$$(126) \frac{1}{2n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m-1} + \frac{1}{2m} \\ - \frac{1}{12} \left(\frac{1}{n^2} - \frac{1}{m^2} \right) + \frac{1}{120} \left(\frac{1}{n^4} - \frac{1}{m^4} \right) - \frac{1}{252} \left(\frac{1}{n^6} - \frac{1}{m^6} \right) \\ + \frac{1}{240} \left(\frac{1}{n^8} - \frac{1}{m^8} \right) + \dots \infty$$

$$(127) \frac{1}{2} \log n + \log (n+1) + \dots + \log (m-1) \\ + \frac{1}{2} \log m - \frac{1}{12} \left(\frac{1}{m} - \frac{1}{n} \right) + \frac{1}{360} \left(\frac{1}{m^3} - \frac{1}{n^3} \right) \\ - \frac{1}{1260} \left(\frac{1}{m^5} - \frac{1}{n^5} \right) + \dots \infty$$

$$(128) \sum_1^{\infty} \left[n \log \left(\frac{2n+1}{2n-1} \right) - 1 \right]$$

$$(129) x - \frac{2}{3} x^3 + \frac{2 \cdot 4}{3 \cdot 5} x^5 + \dots \infty$$

$$(130) 1 + (x\epsilon^{-x}) + \frac{3}{2!} (x\epsilon^{-x})^2 + \frac{4^2}{3!} (x\epsilon^{-x})^3 + \frac{5^3}{4!} (x\epsilon^{-x})^4 + \dots \infty \\ = 1 + \sum_1^{\infty} \frac{(n+1)^{n-1}}{n!} (x\epsilon^{-x})^n$$

$$(131) \frac{x^3}{1 \cdot 3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{5 \cdot 7} + \dots \infty \quad (\text{if convergent})$$

$$(132) \dagger 1 - 2(2-1)B_1 \frac{x^2}{2} + 2(2^3-1)B_2 \frac{x^4}{4!} \\ - 2(2^5-1)B_3 \frac{x^6}{6!} + \dots \infty$$

$$(133) \dagger 1 + 2ax + (2a)^2 \left\{ B_2 \left(\frac{1}{2} + x \right) + \frac{1}{2} B_1 \right\} + \frac{(2a)^3}{2!} B_2 \left(\frac{1}{2} + x \right) \\ + \frac{(2a)^4}{3!} \left\{ B_4 \left(\frac{1}{2} + x \right) - \frac{1}{4} B_2 \right\} + \dots \infty$$

† For values of $B_n(x)$, see No. (1146).

$$= \log_h \frac{m}{n} \quad \text{where } m \text{ and } n \text{ integers} \quad \text{X. 141}$$

$$= m \log_h \frac{m}{\epsilon} - n \log_h \frac{n}{\epsilon} \quad \text{where } m \text{ and } n \text{ integers} \quad \text{X. 141}$$

$$= \frac{1}{2} (1 - \log_h 2) \quad \text{A. 526}$$

$$= \frac{1}{\sqrt{1+x^2}} \log_h \{x + \sqrt{1+x^2}\} \quad \text{where } |x| \leq 1 \quad \text{A. 197}$$

$$= e^x \quad \text{Y. 456}$$

$$= \frac{x^2 - 1}{4} \log_h \frac{1+x}{1-x} + \frac{x}{2} \quad \text{1Z. 165}$$

$$= \frac{2x}{e^x - e^{-x}} \quad \text{AE. 12}$$

$$= \frac{2a\epsilon^{2ax}}{\epsilon^a - \epsilon^{-a}} \quad \text{AE. 14}$$

Series No.

$$(134)^\dagger \frac{1}{2} + xB_2\left(\frac{1}{2}\right) + \frac{x^3}{3!}B_4\left(\frac{1}{2}\right) + \frac{x^5}{5!}B_6\left(\frac{1}{2}\right) + \dots \infty$$

$$(135)^\dagger \frac{1}{3} + 3xB_2\left(\frac{1}{3}\right) + \frac{(3x)^2}{2!}B_3\left(\frac{1}{3}\right) + \frac{(3x)^3}{3!}B_4\left(\frac{1}{3}\right) + \dots \infty$$

$$(136)^\dagger \frac{1}{6} + 6xB_2\left(\frac{1}{6}\right) + \frac{(6x)^2}{2!}B_3\left(\frac{1}{6}\right) + \dots \infty$$

$$(137) \quad x - \left(1 + \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 \\ - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)x^4 + \dots \infty$$

$$(138) \quad x - \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} - \dots \infty$$

$$(139) \quad \log h 2 + \frac{1 \cdot 1}{2 \cdot 2}x^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4}x^4 + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6}x^6 - \dots \infty$$

$$(140) \quad 1 - \frac{x}{2} - \frac{x^2}{12} - \frac{x^3}{24} - \dots \infty = \int_0^1 (1-x)^t dt$$

$$(141) \quad x + \frac{x^3}{3} - \frac{2}{3} \cdot \frac{x^5}{5} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{x^7}{7} - \dots \infty$$

$$(142) \quad x - \frac{2}{3}x^3 + \frac{2 \cdot 4}{3 \cdot 5}x^5 + \dots \infty$$

$$(143) \quad \frac{x^2}{2} - \frac{2}{3} \cdot \frac{x^4}{4} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{x^6}{6} + \dots \infty$$

$$(144) \quad -x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{3} - \frac{x^7}{7} + \frac{x^8}{8} + \dots \infty$$

$$(145) \quad \frac{x^2}{2} + \frac{2x^3}{3} + \frac{3x^4}{4} + \dots \infty$$

† For values of $B_n(x)$, see No. (1146).

	<i>Reference</i>
$= \frac{1}{e^{x/2} + 1}$	AE. 30
$= \frac{1}{1 + e^x + e^{2x}}$	AE. 35
$= \frac{1}{1 + e^x + e^{2x} + e^{3x} + e^{4x} + e^{5x}}$	AE. 43
$= \frac{\log h (1 + x)}{1 + x}$	R. 425
$= \log h (x + \sqrt{1 + x^2})$ where $-1 \leq x \leq 1$	L. 78
$= \log h (1 + \sqrt{1 + x^2})$ where $x^2 < 1$	T. 123
$= \frac{x}{\log h \frac{1}{1 - x}}$	A. 190
$= \sqrt{1 + x^2} \log h \{x + \sqrt{1 + x^2}\}$	A. 191
$= \frac{1}{\sqrt{1 + x^2}} \log h \{x + \sqrt{1 + x^2}\}$ where $x < 1$	A. 197
$= \frac{1}{2} \{\log h (x + \sqrt{1 + x^2})\}^2$	L. 77
$= \log h (1 - x + x^2)$	L. 79
$= \frac{x}{1 - x} + \log h (1 - x)$ where $x < 1$	F. 197

Series No.

$$(146) \log h 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} - \dots \infty$$

$$(147) \dagger \frac{x}{2} - B_1(2^2 - 1) \frac{x^2}{2!} - B_2(2^4 - 1) \frac{x^4}{4!} \\ - B_3(2^6 - 1) \frac{x^6}{6!} - \dots \infty$$

$$(148) \dagger 2 \left\{ B_1(2^2 - 1) \frac{x}{2!} - B_2(2^4 - 1) \frac{x^3}{4!} + B_3(2^6 - 1) \frac{x^5}{6!} - \dots \infty \right\}$$

$$(149) \frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \dots \infty$$

$$(150) x + \frac{x^5}{5!} + \frac{x^9}{9!} + \dots \infty$$

$$(151) \dagger \log h 2 + \log h 3 + \dots \log h (n-1) + \frac{1}{2} \log h n$$

(This series is not convergent.)

$$(152) \dagger 1 - \frac{x}{2} + \frac{B_1^*}{2!} x^2 - \frac{B_3^*}{4!} x^4 + \frac{B_5^*}{6!} x^6 + \dots \infty$$

$$(153) 1 + \frac{x}{2} - \frac{1}{12} x^2 + \frac{1}{24} x^3 - \frac{19}{720} x^4 + \dots \infty$$

$$(154) 1 + \frac{x^2}{2 \cdot 3} - \frac{x^4}{4 \cdot 5} + \frac{x^6}{6 \cdot 7} + \dots \infty = 1 + \sum_1^{\infty} \frac{x^{2n}(-1)^{n-1}}{2n(2n+1)}$$

$$(155) x^3 + x^4 + \frac{x^5}{2} + \dots \infty \quad \text{et puis! quoi?}$$

$$(156) \frac{x^r}{r!} - {}_r P_1 \frac{x^{r+1}}{(r+1)!} + {}_{r+1} P_2 \frac{x^{r+2}}{(r+2)!} - \dots \infty$$

† For B_1, B_2 , etc., see No. (1129).‡ For values of B_1 and B_1^* , etc., see No. (1129).

Reference

$$= \log h (1 + \epsilon^x) \quad \text{H. 498}$$

$$= \frac{x}{\epsilon^x + 1} \quad \text{N. 1543}$$

$$= \frac{\epsilon^x - 1}{\epsilon^x + 1} \quad \text{N. 1544}$$

$$= 1 + \frac{1-x}{x} \log h (1-x) \quad \text{F. 338}$$

$$= \frac{1}{4} (\epsilon^x - \epsilon^{-x} - j\epsilon^{jx} + j\epsilon^{-jx}) \quad \text{where } j = \sqrt{-1} \quad \text{F. 338}$$

$$\approx = \frac{1}{2} \log h (2\pi) + n \log h n - n \quad \text{Stirling}$$

$$+ \frac{B_1}{1 \cdot 2 \cdot n} - \frac{B_2}{3 \cdot 4 \cdot n^3} + \dots + \frac{(-1)^{r-1} B_r}{(2r-1) \cdot 2r \cdot n^{2r-1}} \dots \quad \text{1P. 612}$$

$$= \frac{x}{\epsilon^x - 1} \quad \text{2Z. 123}$$

$$= \frac{x}{\log h (1+x)} \quad ? \quad \text{W. 243}$$

$$= \frac{1}{x} \tan^{-1} x + \frac{1}{2} \log h (1+x^2)$$

$$= \log h (1 + x^3 \epsilon^x) \quad \text{Y. 80}$$

$$= \frac{[\log h (1+x)]^r}{r!} \quad \text{where } P_k \text{ is the sum of all products } k \text{ at a time, of the first } r \text{ natural numbers} \quad \text{Y. 80}$$

Series No.

$$(157) \frac{x}{2} - \frac{x^2}{24} + \frac{x^4}{2880} - \dots \infty$$

$$(158) x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{4x^5}{5} + \frac{x^6}{6} + \dots \infty$$

(159) Reversion of Series.

$$y = x - b_1x^2 - b_2x^3 - b_3x^4 - \dots \infty$$

can become

$$x = y + C_1y^2 + C_2y^3 + C_3y^4 + \dots \infty$$

if

$$C_1$$

$$C_2$$

$$C_3$$

$$C_4$$

$$C_5$$

$$C_6$$

$$C_7$$

See Van Orstrand (*Phil. Mag.* 19: 366.1910) for coefficients up to C_{12} .

$$(160) 1 - \frac{1}{2(n+1)} - \frac{1}{2 \cdot 3(n+1)^2} - \frac{1}{3 \cdot 4(n+1)^3} - \dots \infty$$

$$(161) 1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \dots \infty$$

$$(162) \frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \dots \infty$$

$$(163) \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \dots \infty$$

Reference

$$= \log h \frac{x e^x}{e^x - 1} \quad \text{Y. 107}$$

$$= \log h (1 + x + x^2 + x^3 + x^4) \quad \text{Y. 107}$$

T. 116

$$= b_1$$

$$= b_2 + 2b_1^2$$

$$= b_3 + 5b_1b_2 + 5b_1^3$$

$$= b_4 + 6b_1b_3 + 3b_2^2 + 21b_1^2b_2 + 14b_1^4$$

$$= b_5 + 7(b_1b_4 + b_2b_3) + 28(b_1^2b_3 + b_1b_2^2) + 84b_1^3b_2 + 42b_1^5$$

$$= b_6 + 4(2b_1b_5 + 2b_2b_4 + b_3^2) + 12(3b_1^2b_4 + 6b_1b_2b_3 + b_2^3) \\ + 60(2b_1^3b_3 + 3b_1^2b_2^2) + 330b_1^4b_2 + 132b_1^6$$

$$= b_7 + 9(b_1b_6 + b_2b_5 + b_3b_4) \\ + 45(b_1^2b_5 + b_1b_3^2 + b_2^2b_3 + 2b_1b_2b_4) \\ + 165(b_1^3b_4 + b_1b_2^3 + 3b_1^2b_2b_3) \\ + 495(b_1^4b_3 + 2b_1^3b_2^2) + 1287b_1^5b_2 + 429b_1^7$$

$$= \log h \left\{ 1 + \frac{1}{n} \right\}^n \quad \text{F. 197}$$

$$= 5e \sum_{m=0}^{\infty} \frac{B_m}{m!} = B_K \cdot e \quad \text{F. 197}$$

$$= 2 \log h n - \log h (n+1) - \log h (n-1) \quad \text{F. 197}$$

$$= \log h \frac{n}{n-1} \quad \text{C. 368}$$

Series No.

$$(164) -\frac{x}{2} + \frac{5}{24}x^2 - \frac{1}{8}x^3 + \frac{251}{2880}x^4 - \dots \infty$$

IX. Binomials. See also No. (1102).

$$(165) x^n + nx^{n-1}a + \frac{n(n-1)}{2!}x^{n-2}a^2 + \dots + a^n$$

$$(166) 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \dots \infty$$

$$(167) 2 + \frac{5}{3 \cdot 2!} + \frac{5 \cdot 7}{3^2 \cdot 3!} + \frac{5 \cdot 7 \cdot 9}{3^3 \cdot 4!} + \dots \infty$$

$$(168) 1 - \frac{1}{2} \cdot \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{2^2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{2^3} + \dots \infty$$

$$(169) 1 + \frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12} + \dots \infty$$

$$(170) 1 + \frac{1}{2}x - \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \dots \infty$$

$$(171) 1 - \frac{1}{2}x - \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 - \dots \infty$$

(The above two series are useful in forming certain trigonometrical series.)

$$(172) \frac{1}{3}x + \frac{1 \cdot 4}{3 \cdot 6}2^2x^2 + \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9}3^2x^3 + \dots \infty$$

$$(173) 1 - x + x^2 - x^3 + \dots \infty$$

$$(174) 1 - 2x + 3x^2 - 4x^3 + \dots \infty$$

$$(175) 1 + \frac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \dots \infty$$

$$(176) 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 - \dots \infty$$

Reference

$$= \log h \{ \log h (1 + x)^{1/x} \}$$

Y. 107

$$= (x + a)^n$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

H. 468

$$= 3\sqrt{3}$$

F. 167

$$= \sqrt{2/3}$$

F. 168

$$= \sqrt{8}$$

F. 168

$$= \sqrt{\frac{\sqrt{1 + x^2} + x}{1 + x^2}}$$

$$= \sqrt{\frac{\sqrt{1 + x^2} - x}{1 + x^2}}$$

$$= \frac{x(x + 3)}{9(1 - x)^{7/3}}$$

$$= (1 + x)^{-1}$$

T. 117

$$= (1 + x)^{-2}$$

T. 117

$$= \sqrt{1 + x}$$

T. 117

$$= \frac{1}{\sqrt{1 + x}}$$

T. 117

Series No.

$$(177) \quad 1 + \frac{1}{3}x - \frac{1 \cdot 2}{3 \cdot 6}x^2 + \frac{1 \cdot 2 \cdot 5}{3 \cdot 6 \cdot 9}x^3 - \frac{1 \cdot 2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}x^4 + \dots \infty$$

$$(178) \quad 1 - \frac{1}{3}x + \frac{1 \cdot 4}{3 \cdot 6}x^2 - \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9}x^3 + \frac{1 \cdot 4 \cdot 7 \cdot 10}{3 \cdot 6 \cdot 9 \cdot 12}x^4 - \dots \infty$$

$$(179) \quad 1 + \frac{3}{2}x + \frac{3 \cdot 1}{2 \cdot 4}x^2 - \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6}x^3 + \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8}x^4 - \dots \infty$$

$$(180) \quad 1 - \frac{3}{2}x + \frac{3 \cdot 5}{2 \cdot 4}x^2 - \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6}x^3 + \dots \infty$$

$$(181) \quad 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 - \frac{77}{2048}x^4 + \dots \infty$$

$$(182) \quad 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \frac{195}{2048}x^4 - \dots \infty$$

$$(183) \quad 1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \frac{21}{625}x^4 + \dots \infty$$

$$(184) \quad 1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \frac{44}{625}x^4 - \dots \infty$$

$$(185) \quad 1 + \frac{1}{6}x - \frac{5}{72}x^2 + \frac{55}{1296}x^3 - \frac{935}{31104}x^4 + \dots \infty$$

$$(186) \quad 1 - \frac{1}{6}x + \frac{7}{72}x^2 - \frac{91}{1296}x^3 + \frac{1729}{31104}x^4 - \dots \infty$$

$$(187) \quad 1 + n \left(\frac{x}{4} \right) + \frac{n(n-3)}{2!} \left(\frac{x}{4} \right)^2 + \frac{n(n-4)(n-5)}{3!} \left(\frac{x}{4} \right)^3 + \dots \infty$$

$$(188) \quad 1 + \frac{n^2}{2!}x^2 + \frac{n^2(n^2-2^2)}{4!}x^4 + \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}x^6 + \dots$$

$$+ \frac{n}{1!}x + \frac{n(n^2-1^2)}{3!}x^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 + \dots \infty$$

Reference

$$= (1 + x)^{1/3} \quad \text{T. 117}$$

$$= (1 + x)^{-1/3} \quad \text{T. 117}$$

$$= (1 + x)^{3/2} \quad \text{T. 117}$$

$$= (1 + x)^{-3/2} \quad \text{T. 117}$$

$$= (1 + x)^{1/4} \quad \text{T. 117}$$

$$= (1 + x)^{-1/4} \quad \text{T. 117}$$

$$= (1 + x)^{1/5} \quad \text{T. 118}$$

$$= (1 + x)^{-1/5} \quad \text{T. 118}$$

$$= (1 + x)^{1/6} \quad \text{T. 118}$$

$$= (1 + x)^{-1/6} \quad \text{T. 118}$$

$$= \frac{1}{2^n} \{1 + \sqrt{1 + x}\}^n \quad \text{where } x^2 < 1 \text{ and } n \text{ is any real number}$$

T. 118

$$= \{x + \sqrt{1 + x^2}\}^n \quad \text{where } x^2 < 1 \quad \text{T. 118}$$

Series No.

$$(189) {}_m C_n = \frac{m(m-1)(m-2)\dots(m-n+1)}{n!}$$

$$(190) 1 + {}_m C_1 x + \dots {}_m C_n x^n + \dots {}_m C_m x^m$$

$$(191) {}_{m_1} C_n + {}_{m_2} C_1 \cdot {}_{m_1} C_{n-1} + {}_{m_2} C_2 \cdot {}_{m_1} C_{n-2} + {}_{m_2} C_n$$

$$(192) 1 + {}_m C_1 + {}_m C_2 + \dots {}_m C_n + \dots$$

$$(193) 1 - {}_m C_1 + {}_m C_2 \dots (-1)^n {}_m C_n + \dots$$

$$(194) 1 \cdot {}_m C_1 + 2 \cdot {}_m C_2 x + \dots n \cdot {}_m C_n x^{m-1} + \dots$$

$$(195) m(m-1) + \frac{m(m-1)(m-2)}{1!} + \dots \\ + \frac{m(m-1)\dots(m-r+1)}{(r-2)!} + \dots$$

$$(196) 1 + m + \frac{m(m+1)}{2!} + \dots + \frac{m(m+1)\dots(m+r-1)}{r!}$$

$$(197) 2^n \left\{ x^n + \frac{n}{1 \cdot 2^2} x^{n-2} y^2 + \frac{n(n-3)}{2! 2^4} x^{n-4} y^4 + \dots \right. \\ \left. + \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r! 2^{2r}} x^{n-2r} y^{2r} + \dots \right\}$$

$$(198) 2^n (x^2 + y^2)^{1/2} \\ \times \left\{ x^{n-1} + \frac{n-2}{1! 2^2} x^{n-3} y^2 + \frac{(n-3)(n-4)}{2! 2^4} x^{n-5} y^4 + \dots \right. \\ \left. + \frac{(n-r-1)(n-r-2)\dots(n-2r)}{r! 2^{2r}} x^{n-2r-1} y^{2r} + \dots \right\}$$

$$(199) 1 - {}_m C_1 + {}_m C_2 - \dots (-1)^n {}_m C_n$$

$$(200) {}_n C_1 - \frac{1}{2} {}_n C_2 + \frac{1}{3} {}_n C_3 - \dots$$

Reference

C. 186

$$= (1 + x)^m \quad \text{C. 186}$$

$$= {}_{m_1+m_2}C_n \quad \text{C. 189}$$

$$= 2^m \quad \text{where } m > -1 \quad \text{C. 191}$$

$$= 0 \quad \text{where } m \text{ is positive} \quad \text{C. 191}$$

$$= m(1 + x)^{m-1} \quad \text{C. 197}$$

$$= m(m-1)2^{m-2} \quad \text{where } m \leq 1 \quad \text{C. 200}$$

$$= \frac{(m+r)!}{m!r!} \quad \text{where } r \text{ is a positive integer} \quad \text{C. 200}$$

$$= (x + \sqrt{x^2 + y^2})^n + \{x - \sqrt{x^2 + y^2}\}^n \quad \text{where } n \text{ is a positive integer} \quad \text{C. 204}$$

$$= \{x + \sqrt{x^2 + y^2}\}^n - \{x - \sqrt{x^2 + y^2}\}^n \quad \text{where } n \text{ is a positive integer} \quad \text{C. 205}$$

$$= (-1)^n {}_n C_{m-1} \quad \text{where } n \text{ is a positive integer} \quad \text{C. 210}$$

$$= 1 + \frac{1}{2} + \dots + \frac{1}{n} \quad \text{C. 212}$$

*Series No.***X. Simple Inverse Products**

(201)† To find the sum of n terms of a series, each term of which is composed of the reciprocal of the product of r factors in arithmetical progression, the first factors of the several terms being in the same arithmetical progression: Write down the n th term, strike off a factor from the beginning, divide by the number of factors so diminished, and by the common difference change the sign and add a constant.

$$(202) \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \dots n \text{ terms}$$

$$(203) \frac{3}{1 \cdot 2 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 5} + \frac{5}{3 \cdot 4 \cdot 6} + \dots n \text{ terms}$$

$$(204) \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots n \text{ terms} \\ + \dots \infty$$

$$(205) \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots n \text{ terms}$$

$$(206) \frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} \cdot 2 + \frac{3}{4 \cdot 5} \cdot 2^2 + \dots n \text{ terms}$$

$$(207) \frac{2}{1 \cdot 3 \cdot 4} + \frac{3}{2 \cdot 4 \cdot 5} + \dots n \text{ terms}$$

$$(208) \frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \dots n \text{ terms} \\ + \dots \infty = \sum_1^{\infty} \frac{1}{(3n-2)(3n+1)}$$

† In some cases the n th term can by partial fractions be resolved into the standard form when this rule can apply.

F. 316

$$= \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)} \quad \text{F. 317}$$

$$= \frac{29}{36} - \frac{1}{n+3} - \frac{3}{2(n+2)(n+3)} - \frac{4}{3(n+1)(n+2)(n+3)} \quad \text{F. 317}$$

$$= \frac{n}{n+1} \quad \text{F. 322}$$

$$= 1$$

$$= \frac{n}{2n+1}$$

$$= \frac{2^n}{n+2} - \frac{1}{2}$$

$$= \frac{17}{36} - \frac{6n^2 + 21n + 17}{6(n+1)(n+2)(n+3)}$$

$$= \frac{n}{3n+1}$$

$$= \frac{1}{3} \quad \text{--- OK } \left[\sum \phi(4/3, 8, 6)^{-1} = 1 \right] \quad \text{F. 322}$$

Series No.

$$(209) \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \dots n \text{ terms} \\ + \dots \infty$$

$$(210) \frac{1}{1 \cdot 4 \cdot 7} + \frac{1}{4 \cdot 7 \cdot 10} + \frac{1}{7 \cdot 10 \cdot 13} + \dots n \text{ terms} \\ + \dots \infty$$

$$(211) \frac{4}{1 \cdot 2 \cdot 3} + \frac{5}{2 \cdot 3 \cdot 4} + \frac{6}{3 \cdot 4 \cdot 5} + \dots n \text{ terms} \\ + \dots \infty$$

$$(212) \frac{1}{3 \cdot 4 \cdot 5} + \frac{2}{4 \cdot 5 \cdot 6} + \frac{3}{5 \cdot 6 \cdot 7} + \dots n \text{ terms} \\ + \dots \infty$$

$$(213) \frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \dots n \text{ terms} \\ + \dots \infty$$

$$(214) \frac{3}{1 \cdot 2} \cdot \frac{1}{2} + \frac{4}{2 \cdot 3} \cdot \frac{1}{2^2} + \frac{5}{3 \cdot 4} \cdot \frac{1}{2^3} + \dots n \text{ terms}$$

$$(215) \frac{1^2}{2 \cdot 3} \cdot 4 + \frac{2^2}{3 \cdot 4} \cdot 4^2 + \frac{3^2}{4 \cdot 5} \cdot 4^3 + \dots n \text{ terms}$$

$$(216) \frac{1}{1 \cdot 3} + \frac{2}{1 \cdot 3 \cdot 5} + \frac{3}{1 \cdot 3 \cdot 5 \cdot 7} + \dots n \text{ terms}$$

$$(217) \frac{1 \cdot 2}{3!} + \frac{2 \cdot 2^2}{4!} + \frac{3 \cdot 2^3}{5!} + \dots n \text{ terms}$$

$$(218) \frac{1 \cdot 2}{3} + \frac{2 \cdot 3}{3^2} + \frac{3 \cdot 4}{3^3} + \dots \infty$$

$$(219) \frac{5}{1 \cdot 2} \cdot \frac{1}{3} + \frac{7}{2 \cdot 3} \cdot \frac{1}{3^2} + \frac{9}{3 \cdot 4} \cdot \frac{1}{3^3} + \dots n \text{ terms}$$

Reference

$$= \frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$$

$$= \frac{1}{12} \quad \text{F. 322}$$

$$= \frac{1}{24} - \frac{1}{6(3n+1)(3n+4)}$$

$$= \frac{1}{24} \quad \text{F. 322}$$

$$= \frac{5}{4} - \frac{2n+5}{2(n+1)(n+2)}$$

$$= \frac{5}{4} \quad \text{F. 322}$$

$$= \frac{1}{6} - \frac{1}{n+3} + \frac{2}{(n+3)(n+4)}$$

$$= \frac{1}{6} \quad \text{F. 322}$$

$$= \frac{3}{4} - \frac{2}{n+2} + \frac{1}{2(n+1)(n+2)}$$

$$= \frac{3}{4} \quad \text{F. 322}$$

$$= 1 - \frac{1}{n+1} \cdot \frac{1}{2^n} \quad \text{F. 333}$$

$$= \frac{n-1}{n+2} \cdot \frac{4^{n+1}}{3} + \frac{2}{3} \quad \text{F. 333}$$

$$= \frac{1}{2} - \frac{1}{2} \frac{1}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)} \quad \text{F. 333}$$

$$= 1 - \frac{2^{n+1}}{(n+2)!} \quad \text{F. 333}$$

$$= \frac{9}{4} \quad \text{F. 332}$$

$$= 1 - \frac{1}{n+1} \cdot \frac{1}{3^n} \quad \text{F. 331}$$

Series No.

$$(220) \frac{1}{3} + \frac{3}{3 \cdot 7} + \frac{5}{3 \cdot 7 \cdot 11} + \dots n \text{ terms}$$

$$(221) \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \dots \frac{1}{n(n+2)}$$

$$(222) \sum_1^n \frac{1}{(1+nx)(1+n+1x)}$$

$$(223) \frac{1}{(x+1)(x+2)} + \frac{1!}{(x+1)(x+2)(x+3)} + \frac{2!}{(x+1)(x+2)(x+3)(x+4)} + \dots \infty$$

$$(224) \sum_1^\infty \frac{1}{(x+n)(x+n+1)} \quad \text{deleted}$$

$$(225) \frac{1}{a} + \frac{x}{a(a+1)} + \frac{x^2}{a(a+1)(a+2)} + \dots \infty$$

$$(226) \frac{a}{b} + \frac{a(a+1)}{b(b+1)}x + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^2 + \dots \infty$$

$$(227) \frac{1}{2n+2} + \frac{1}{2} \cdot \frac{1}{2n+4} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{2n+6} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{2n+8} + \dots \infty$$

$$(228) 1 + \frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \dots \infty$$

Reference

$$= \frac{1}{2} - \frac{1}{2} \frac{1}{3 \cdot 7 \cdot 11 \dots (4n-1)}$$

F. 331

$$= \frac{3n^2 + 5n}{4(n+1)(n+2)} = 3/4$$

W. 57

$$= \frac{n}{(1+x)\{1+n+1x\}}$$

$$= \frac{1}{(1+x)^2}$$

$$= \frac{1}{x+1}$$

$\phi(2,4,2) \xrightarrow{\text{elem}} \phi(M_2) = 1/n$

$$= \frac{(a-1)!}{x^a} \left\{ e^x - \sum_{n=0}^{a-1} \frac{x^n}{n!} \right\} \quad \text{where } a \text{ is positive}$$

T. 118

$$= (b-a) \binom{b-1}{a-1} \left\{ \frac{(-1)^{b-a} \log(1-x)}{x^b} (1-x)^{b-a-1} + \frac{1}{x^a} \sum_{k=1}^{b-a} (-1)^k \binom{b-a-1}{k-1} \sum_{n=1}^{a+k-1} \frac{x^{n-k}}{n} \right\}$$

where a and b are positive and $a < b$; $\binom{b-1}{a-1}$, etc., are binomial coefficients

T. 118

$$= \frac{2 \cdot 4 \cdot 6 \dots 2n}{3 \cdot 5 \cdot 7 \dots (2n+1)}$$

1Z. 267

$$= \frac{b-1}{b-a-1} \quad \text{where } b-1 > a > 0$$

A. 48

Series No.

$$(229) \frac{a}{b} + \frac{2a(a+1)}{b(b+1)} + \frac{3a(a+1)(a+2)}{b(b+1)(b+2)} + \dots \infty$$

$$(230) \frac{1}{x(x+1)} + \frac{1}{x(x+1)(x+2)} + \frac{1 \cdot 2}{x(x+1)(x+2)(x+3)} + \dots \infty$$

$$(231) \frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \dots n \text{ terms}$$

$$(232) \frac{1}{(1+x)(1+ax)} + \frac{a}{(1+ax)(1+a^2x)} + \dots n \text{ terms}$$

XI. Other Inverse Products

$$(233) \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots \infty$$

$$(234) \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 6} + \frac{1}{4 \cdot 8} + \dots \infty$$

$$(235) \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \dots \infty$$

$$(236) \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} + \dots \infty$$

$$(237) \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} - \dots \infty$$

$$(238) \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{5 \cdot 7 \cdot 9} + \frac{1}{9 \cdot 11 \cdot 13} + \dots \infty$$

$$(239) \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \dots \infty$$

$$(240) \frac{1}{1 \cdot 3 \cdot 5} - \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} - \dots \infty$$

$$(241) \left(\frac{1}{1 \cdot 2 \cdot 3} \right)^2 + \left(\frac{1}{2 \cdot 3 \cdot 4} \right)^2 + \left(\frac{1}{3 \cdot 4 \cdot 5} \right)^2 + \dots \infty$$

Reference

$$= \frac{a(b-1)}{(b-a-1)(b-a-2)} \quad \text{where } b-2 > a > 0$$

A. 48

$$= \frac{1}{x^2}$$

$$= \frac{n}{(1+x)(1+x.n+1)}$$

F. 313

$$= \frac{1}{1-a} \left(\frac{1}{1+x} - \frac{a^n}{1+a^n x} \right)$$

F. 313

$$= \frac{1}{2}$$

T. 143

$$= \frac{\pi^2}{12}$$

0,8224670337

E. 158

$$\log(1) - 1 = \log_h \frac{4}{e}$$

~~0,1281604~~ 0,3862943613 C. 252

$$= \log_h 2 - \frac{1}{2} = 0.1931471806$$

F. 338

$$= 0.153426 = \frac{1}{2}(1 - \log_h 2) \quad 0,1534264097 \quad \text{H. 476}$$

$$= \frac{\pi}{16} - \frac{1}{8}$$

0,07134954090

$$= \frac{1}{12}$$

0,083

$$= \frac{\pi}{8} - \frac{1}{3}$$

0,05936574850

C. 372

$$= \frac{\pi^2}{4} - \frac{39}{16}$$

0,02990110100

O. 370

Series No.

$$(242) \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7 \cdot 8} + \frac{1}{9 \cdot 10 \cdot 11 \cdot 12} + \dots \infty$$

$$(243) \frac{1}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{1}{9 \cdot 11 \cdot 13 \cdot 15} + \frac{1}{17 \cdot 19 \cdot 21 \cdot 23} + \dots \infty$$

$$(244) \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{6 \cdot 7 \cdot 8} - \dots \infty$$

$$(245) \frac{5}{1 \cdot 2} - \frac{3}{2 \cdot 3} + \frac{9}{3 \cdot 4} - \frac{7}{4 \cdot 5} + \frac{13}{5 \cdot 6} - \frac{11}{6 \cdot 7} + \dots n \text{ terms}$$

$$(246) \frac{2}{1 \cdot 5} - \frac{4}{5 \cdot 7} + \frac{8}{7 \cdot 17} - \frac{16}{17 \cdot 31} + \frac{32}{31 \cdot 65} - \dots n \text{ terms}$$

$$(247) \frac{1}{(1 \cdot 3)^2} + \frac{1}{(3 \cdot 5)^2} + \frac{1}{(5 \cdot 7)^2} + \dots \infty$$

$$(248) \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \dots n \text{ terms}$$

$$(249) \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{6 \cdot 7 \cdot 8} + \dots \infty$$

$$(250) \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{7 \cdot 8 \cdot 9} + \dots \infty$$

$$(251) \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{6 \cdot 7 \cdot 8} + \frac{1}{10 \cdot 11 \cdot 12} + \dots \infty$$

$$(252) \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{4 \cdot 5 \cdot 6 \cdot 7} + \frac{1}{7 \cdot 8 \cdot 9 \cdot 10} + \dots \infty$$

$$(253) \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{5 \cdot 6 \cdot 7} + \frac{1}{9 \cdot 10 \cdot 11} + \dots \infty$$

$$(254) 1 - \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \dots \infty$$

$$(255) 1 + \frac{1}{2} - \frac{1 \cdot 1}{2 \cdot 4} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \dots \infty$$

$= \frac{1}{4} \log h 2 - \frac{\pi}{24}$	0,0423871013	<i>Reference</i> O. 371
$= \frac{\pi}{96(2 + \sqrt{2})}$	0,009584908174	O. 371
$= \frac{1}{4} (\pi - 3)$	0,03539816340	A. 190
$= 3 - \frac{2 + (-1)^n}{n + 1}$		F. 339
$= \frac{1}{3} \left\{ 1 + \frac{(-1)^{n+1}}{2^{n+1} + (-1)^{n+1}} \right\}$		F. 339
$= \frac{\pi^2 - 8}{16}$	0,1168502753	
$= \frac{1}{2} - \frac{1 \cdot 3 \cdot 5 \dots (2n + 1)}{2 \cdot 4 \cdot 6 \dots (2n + 2)}$		F. 333
$= \frac{3}{4} - \log h 2$	0,05685281940	T. 144
$= \frac{1}{4} \left(\frac{\pi}{\sqrt{3}} - \log h 3 \right)$	0,1781967688	T. 144
$= \frac{\pi}{8} - \frac{1}{2} \log h 2$	0,04612549150	T. 144
$= \frac{1}{6} \left(1 + \frac{\pi}{2\sqrt{3}} \right) - \frac{1}{4} \log h 3$	0,04316354140	T. 144
$= \frac{1}{4} \log h 2$	0,1732867952	C. 252
$= \frac{1}{\sqrt{2}}$		
$= \sqrt{2}$		

Series No.

$$(256) \frac{1}{3} + \frac{2}{3 \cdot 5} + \frac{3}{3 \cdot 5 \cdot 7} + \frac{4}{3 \cdot 5 \cdot 7 \cdot 9} + \dots \infty$$

$$(257) \frac{1}{3} + \frac{2}{3 \cdot 5} + \frac{3}{3 \cdot 5 \cdot 7} + \frac{4}{3 \cdot 5 \cdot 7 \cdot 9} + \dots n \text{ terms}$$

$$(258) 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \dots \infty$$

$$(259) 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \dots \infty$$

$$= 1 + \frac{2}{3} \cdot \frac{1}{2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{1}{2^2} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{1}{2^3} + \dots$$

$$(260) 1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \dots \infty$$

$$(261) 1 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1 \cdot 2}{3 \cdot 5} \cdot \frac{1}{2^2} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} \cdot \frac{1}{2^3} + \dots \infty$$

$$(262) \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{3} + \dots \infty$$

$$(263) \frac{1}{2 \cdot 5} + \frac{1}{8 \cdot 11} + \frac{1}{14 \cdot 17} + \dots \infty$$

$$(264) 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \dots \infty$$

$$(265) \frac{1}{1 \cdot 5} + \frac{1}{2 \cdot 7} + \frac{1}{3 \cdot 9} + \dots \infty$$

$$(266) \frac{1}{1 \cdot 5} + \frac{1}{9 \cdot 13} + \frac{1}{17 \cdot 21} + \frac{1}{25 \cdot 29} + \dots \infty$$

$$(267) 1 - \frac{2}{3} \cdot \frac{1}{2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{1}{2^2} - \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{1}{2^3} + \dots \infty$$

$$(268) \sum_{t=1}^{\infty} \frac{2}{(t+n-1)(t+n)(t+n+1)}$$

Reference

$$= \frac{1}{2}$$

C. 225

$$= \frac{1}{2} \left\{ 1 - \frac{1}{3 \cdot 5 \cdot 7 \dots (2n+1)} \right\}$$

A. 431

$$= \frac{\pi}{2}$$

Y. 505

$$= \frac{\pi}{2}$$

A. 197

$$= \frac{\pi}{2}$$

A. 184

$$= \frac{2\pi}{3\sqrt{3}}$$

1, 209199576

Y. 505

$$= 2 \log 2$$

1,3862943613

1Z. 136

$$= \frac{1}{9} \left(\frac{\pi}{\sqrt{3}} - \log 2 \right)$$

0,1245169092

1Z. 164

$$= \frac{\pi}{8} (1 + \sqrt{2})$$

0,948059449

1Z. 165

$$= \frac{8}{9} - \frac{2}{3} \log 2$$

0,4267907685

1Z. 165

$$= \frac{\sqrt{2}}{32} \{ \pi + \log (3 + 2\sqrt{2}) \}$$

0,2167432468

1Z. 165

$$= \frac{2}{\sqrt{3}} \log \frac{1 + \sqrt{3}}{\sqrt{2}}$$

A. 197

$$= \frac{1}{i(t+1)}$$

élem

A. 52

Séries $L(1, \chi)$

Dirichlet.

Series No.

$$(269) \sum_{t=1}^{\infty} \frac{3}{(t+n-1)(t+n)(t+n+1)(t+n+2)}$$

$$(270) \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \dots \infty$$

$$(271) \frac{1}{1 \cdot 2^2 \cdot 3} + \frac{1}{5 \cdot 6^2 \cdot 7} + \frac{1}{9 \cdot 10^2 \cdot 11} + \dots \infty$$

$$(272) \frac{1}{1^2 \cdot 2} + \frac{1}{2^2 \cdot 3} + \frac{1}{3^2 \cdot 4} + \dots \infty$$

$$(273) 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \dots \infty$$

$$(274) 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}\right)^2 + \dots \infty$$

$$(275) 1 + \frac{1}{3} \left(\frac{1}{2}\right) + \frac{1 \cdot 2}{3 \cdot 5} \left(\frac{1}{2}\right)^2 + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} \left(\frac{1}{2}\right)^3 + \dots \infty$$

$$(276) 1 + \frac{1}{2 \cdot 3} + \frac{1}{3} \cdot \frac{1 \cdot 2}{3 \cdot 5} + \frac{1}{4} \cdot \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \dots \infty$$

$$(277) 1 + \frac{1}{2} \cdot \frac{1}{3} \left(\frac{1}{2}\right) + \frac{1}{3} \cdot \frac{1 \cdot 2}{3 \cdot 5} \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} \left(\frac{1}{2}\right)^3 + \dots \infty$$

$$(278) 1 + \frac{3}{6} + \frac{3 \cdot 6}{6 \cdot 10} + \frac{3 \cdot 6 \cdot 9}{6 \cdot 10 \cdot 14} + \dots \infty$$

$$(279) \frac{1}{1 \cdot 2 \cdot 3} + \frac{x}{2 \cdot 3 \cdot 4} + \frac{x^2}{3 \cdot 4 \cdot 5} + \dots \infty$$

$$(280) \frac{1}{1 \cdot 2 \cdot 3} + \frac{x}{3 \cdot 4 \cdot 5} + \frac{x^2}{5 \cdot 6 \cdot 7} + \dots \infty$$

$$(281) \frac{1}{1 \cdot 2 \cdot 3} - \frac{x}{3 \cdot 4 \cdot 5} + \frac{x^2}{5 \cdot 6 \cdot 7} - \dots \infty$$

Reference

$$= \frac{1}{t(t+1)(t+2)}$$

o(m)

A. 52

$$= \frac{\pi}{6}$$

ARL SIN(1/2)

H. 468

$$= \frac{\pi}{8} \left(1 - \frac{\pi}{4}\right)$$

D. 495

$$= \frac{\pi^2 - 6}{6}$$

C. 372

$$= \frac{\pi}{2\sqrt{3}}$$

H. 476

$$= \frac{\Gamma(2)}{[\Gamma(3/2)]^2} = \frac{4}{\pi}$$

A. 190

$$= \frac{2\pi}{3\sqrt{3}}$$

L. 236

$$= \frac{\pi^2}{8}$$

L. 237

$$= \frac{\pi^2}{9}$$

L. 237

$$= \frac{4\pi}{3\sqrt{3}}$$

L. 237

$$= \frac{3}{4x} - \frac{1}{2x^2} + \frac{(1-x)^2}{2x^3} \log \frac{1}{1-x} \quad \text{where } x^2 < 1 \quad \text{T. 125}$$

$$= \frac{1}{4x} \left\{ \frac{1+x}{\sqrt{x}} \log \frac{1+\sqrt{x}}{1-\sqrt{x}} + 2 \log (1-x) - 2 \right\} \quad \text{where}$$

$$0 < x < 1 \quad \text{T. 125}$$

$$= \frac{1}{2x} \left\{ 1 - \log (1+x) - \frac{1-x}{\sqrt{x}} \tan^{-1} x \right\} \quad \text{where } 0 < x \leq 1$$

$$\text{T. 125}$$

*Series No.***XII. Simple Factorials**

$$(282) \quad \frac{1}{3!} + \frac{5}{4!} + \frac{11}{5!} + \dots \quad n \text{ terms}$$

$$(283) \quad \frac{2}{1!} + \frac{12}{2!} + \frac{28}{3!} + \frac{50}{4!} + \frac{78}{5!} + \dots \quad \infty$$

$$(284) \quad \frac{1}{n!} + \frac{2!}{(n+1)!} + \frac{3!}{(n+2)!} + \dots \quad \infty$$

$$(285) \quad \frac{0!}{n!} + \frac{1!}{(n+1)!} + \frac{2!}{(n+2)!} + \dots \quad \infty$$

$$(286) \quad \sum_0^{\infty} \frac{(2n)!}{2^{2n} n! n!} \cdot \frac{1}{x+n}$$

$$(287) \quad \sum_{n=0}^{\infty} \frac{(-1)^n a(a-1)(a-2)\dots(a-n)}{n!} \cdot \frac{1}{x+n}$$

$$(288) \quad \sum_1^n n(n!)$$

$$(289) \quad \sum_1^{\infty} (-1)^{n+1} \frac{1}{n(n!)}$$

$$(290) \quad \frac{1}{m} - \frac{n}{1!} \frac{1}{m+1} + \frac{n(n-1)}{2!} \frac{1}{m+2} - \dots$$

$$(291) \quad \frac{1}{3!} + \frac{5}{4!} + \frac{11}{5!} + \dots \frac{n^2 + n - 1}{(n+2)!}$$

XIII. Other Power Series (Bernoulli's and Euler's Numbers)

$$(292) \quad \text{Lt}_{n=\infty} \left[\frac{1^2}{n^3 + 1^3} + \frac{2^2}{n^3 + 2^3} + \frac{3^2}{n^3 + 3^3} + \dots + \frac{n^2}{n^3 + n^3} \right]$$

$$(293) \quad \text{Lt}_{n=\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n} \right]$$

$$= \frac{1}{2} - \frac{n+1}{(n+2)!} \quad \text{F. 333}$$

$$= 5e + 2 \quad \text{F. 334}$$

$$= \frac{1}{(n-2)(n-1)!} \quad \text{F. 338}$$

$$= \frac{1}{(n-1)(n-1)!} \quad \text{AC. 63}$$

$$= \frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})} \quad \text{Q. 259}$$

$$= \frac{\Gamma(x)\Gamma(a+1)}{\Gamma(x+a)} \quad \text{Q. 260}$$

$$= (n+1)! - 1$$

$$= 0.7965996 = \int_0^1 \frac{1-e^{-x}}{x} dx \quad *$$

$$= \frac{(m-1)!}{(n+1)(n+2)\dots(n+m)} \quad \text{C. 211}$$

$$= \frac{1}{2} - \frac{1}{(n+2)(n!)} \quad \text{He. Ho!}$$

$$= \frac{1}{3} \log h 2 \quad \text{1Z. 324}$$

$$= \log h 2 \quad \text{1Z. 326}$$

Series No.

$$(294) \quad \text{Lt}_{n=\infty} \left[\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \dots + \frac{n}{n^2 + n^2} \right]$$

$$(295) \quad \text{Lt}_{n=\infty} \left[\frac{1}{\sqrt{2n-1^2}} + \frac{1}{\sqrt{4n-2^2}} + \frac{1}{\sqrt{6n-3^2}} + \dots + \frac{1}{\sqrt{2n^2-n^2}} \right]$$

$$(296) \quad \text{Lt}_{n=\infty} \left[\frac{(n-m)^{1/3}}{n} + \frac{(2^2n-m)^{1/3}}{2n} + \frac{(3^2n-m)^{1/3}}{3n} + \dots + \frac{(n^3-m)^{1/3}}{n^2} \right]$$

$$(297) \quad \text{Lt}_{n=\infty} \left[\frac{\sqrt{n-1}}{n} + \frac{\sqrt{2n-1}}{2n} + \frac{\sqrt{3n-1}}{3n} + \dots + \frac{\sqrt{n^2-1}}{n^2} \right]$$

$$(298) \quad \text{Lt}_{n=\infty} \left[\frac{1}{\sqrt{2a^2n-1}} + \frac{1}{\sqrt{4a^2n-1}} + \dots + \frac{1}{\sqrt{2a^2n^2-1}} \right]$$

$$(299) \quad \text{Lt}_{n=\infty} \left[\frac{n^2}{(n^2+1^2)^{3/2}} + \frac{n^2}{(n^2+2^2)^{3/2}} + \dots + \frac{n^2}{\{n^2+(n-1)^2\}^{3/2}} \right]$$

$$(300) \quad \text{Lt}_{n=\infty} \left[\frac{\sqrt{n-a}}{n-c} + \frac{\sqrt{2n-a}}{2n-c} + \dots + \frac{\sqrt{n^2-a}}{n^2-c} \right]$$

$$(301) \quad \frac{1}{n+1} + \frac{2}{n^2+1} + \frac{4}{n^4+1} + \dots \infty$$

$$(302) \quad \sum_1^{\infty} \frac{n-1}{n!}$$

$$(302); \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{30} + \frac{1}{144} + \frac{1}{840} + \frac{1}{5760} + \frac{1}{45360} + \frac{1}{403200} + \frac{1}{3991680} + \frac{(n-1)}{n!} + \dots$$

Reference

$$= \frac{\pi}{4}$$

1Z. 326

$$= \frac{\pi}{2}$$

1Z. 326

$$= \frac{3}{2}$$

1Z. 326

$$= 2$$

1Z. 326

$$= \frac{\sqrt{2}}{a}$$

1Z. 326

$$= \frac{1}{\sqrt{2}}$$

1Z. 326

$$= 2$$

1Z. 354

$$= \frac{1}{n-1} \quad \text{where } |n| > 1$$

A. 66

$$= 1$$

A. 52

Series No.

$$(303) \left(n + \frac{1}{2} \right) \log h n - n + \frac{1}{2} \log h (2\pi) + \frac{1}{12n} - \frac{1}{360n^3} - \dots \infty$$

$$(304) n^{n+1/2} \sqrt{2\pi} e^{-n} \left\{ 1 + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \dots \infty \right\}$$

n large

On the series Nos. (305) to (330), see No. (1130) for values of α , β , etc.; see No. (330) for general note covering Nos. (305) through (329).

$$(305) 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots \infty$$

$$(306) 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \dots \infty$$

$$(307) 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \dots \infty$$

$$(308)^\dagger 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \dots \infty$$

$$(309) 1 + \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \dots \infty$$

$$(310) 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \dots \infty$$

$$(311) 1 + \frac{1}{2^n} - \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} - \dots \infty$$

$$^\dagger u_1 = \frac{\pi}{4}$$

$$u_2 = 0.91596\ 56 \dots$$

$$u_4 = 0.98894\ 455 \dots$$

$$u_6 = 0.99868\ 522.$$

A table of n from 1 to 38 to 18 decimal places is given by Glaisher, *Messenger of Mathematics*, 42; 49, 1913.

Reference

$$= \log h(n!)$$

G. 140

$$= n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

G. 140

$$= S_n, \quad S_{2n} = \frac{\frac{1}{2}(2\pi)^{2n} B_n}{(2n)!}$$

AC. 42

$$= s_n, \quad s_{2n} = \frac{\pi^{2n} \beta_n}{2 \cdot (2n)!}$$

AC. 42

$$= U_n, \quad U_{2n} = \frac{\pi^{2n} \alpha_n}{4 \cdot (2n)!}$$

AC. 42

$$= u_n, \quad u_{2n+1} = \frac{\left(\frac{\pi}{2}\right)^{2n+1} E_n^*}{2(2n)!}$$

AC. 42

$$= G_n, \quad G_{2n} = \frac{\frac{2}{3}(\frac{2}{3}\pi)^{2n} \gamma_n}{(2n)!}$$

AC. 42

$$= g_n, \quad \sqrt{3}g_{2n+1} = \frac{(\frac{2}{3}\pi)^{2n+1} I_n}{(2n)!}$$

AC. 42

$$= j_n, \quad \sqrt{3}j_{2n+1} = \frac{(\frac{1}{3}\pi)^{2n+1} J_n}{(2n)!}$$

AC. 42

T. 140

Series No.

$$(312) \quad 1 - \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} - \dots \infty$$

$$(313) \quad 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \dots \infty$$

$$(314) \quad 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \dots \infty$$

$$(315) \quad 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \dots \infty$$

$$(316) \quad 1 - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} - \dots \infty$$

$$(317) \quad 1 + \frac{1}{3^n} - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} - \dots \infty$$

$$(318) \quad 1 - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} - \dots \infty$$

$$(319) \quad 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \dots \infty$$

$$(320) \quad 1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \dots \infty$$

$$(321) \quad 1 - \frac{1}{2^{2n+1}} + \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \dots \infty$$

$$(322) \quad 1 + \frac{1}{2^{2n}} - \frac{2}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} - \frac{2}{6^{2n}} + \dots \infty$$

$$(323) \quad 1 + \frac{1}{2^{2n+1}} - \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \dots \infty$$

$$(324) \quad 1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \dots \infty$$

Reference

$$= v_n, \quad v_{2n} = \frac{\frac{4}{3}(\frac{1}{3}\pi)^{2n}\eta_n}{(2n)!} \quad \text{AC. 42}$$

$$= W_n, \quad W_{2n} = \frac{2(\frac{1}{3}\pi)^{2n}\theta_n}{(2n)!} \quad \text{AC. 42}$$

$$= h_n, \quad \sqrt{3}h_{2n+1} = \frac{(\frac{1}{3}\pi)^{2n+1}H_n}{(2n)!} \quad \text{AC. 42}$$

$$= r_n, \quad r_{2n+1} = \frac{2(\frac{1}{6}\pi)^{2n+1}R_n}{(2n)!} \quad \text{AC. 42}$$

$$= t_n, \quad \sqrt{3}t_{2n} = \frac{6(\frac{1}{6}\pi)^{2n}T_n}{(2n-1)!} \quad \text{AC. 42}$$

$$= p_n, \quad p_{2n+1} = \frac{\sqrt{2}(\frac{1}{4}\pi)^{2n+1}P_n}{(2n)!} \quad \text{AC. 42}$$

$$= q_n, \quad q_{2n} = \frac{\sqrt{2}(\frac{1}{4}\pi)^{2n}Q_n}{(2n-1)!} \quad \text{AC. 42}$$

$$= \frac{(2\pi)^{2n}}{2(2n)!} B_n \quad \text{AE. 3}$$

$$= \frac{2^{2n-1}\pi^{2n}}{(2n)!} \left(1 - \frac{1}{2^{2n-1}}\right) B_n \quad \text{AE. 26}$$

$$= \frac{(-1)^{n+1}(2\pi)^{2n+1}}{\sqrt{3}(2n)!} B_{2n+1} \left(\frac{1}{3}\right) \quad \text{AE. 34}$$

$$= \frac{(-1)^n(2\pi)^{2n}}{(2n-1)!} A_{2n} \left(\frac{1}{3}\right) \quad \text{AE. 34}$$

$$= \frac{(-1)^{n+1}(2\pi)^{2n+1}}{\sqrt{3}(2n)!} B_{2n+1} \left(\frac{1}{6}\right) \quad \text{AE. 40}$$

$$= \frac{(-1)^{n+1}2^{2n}\pi^{2n+1}}{(2n)!} B_{2n+1} \left(\frac{1}{4}\right) \quad \text{AE. 30}$$

$$= \frac{\pi^{2n+1}}{2^{2n+2}(2n)!} E_n^* \quad \text{AE. 30}$$

Series No.

$$(325) \frac{1}{2^{2n}} - \frac{1}{4^{2n}} + \frac{1}{6^{2n}} - \frac{1}{8^{2n}} + \dots \infty$$

$$(326) 1 + \frac{1}{3^{2n+1}} - \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} + \dots \infty$$

$$(327) 1 - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} - \dots \infty$$

$$(328) 1 + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} - \frac{1}{11^{2n+1}} + \dots \infty$$

$$(329) 1 - \frac{1}{5^{2n}} - \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \dots \infty$$

(330) General note on Nos. (305) through (329):

- (a) The values of Bernoulli's and Euler's numbers are given in Nos. (1129) and (1131).
- (b) The values of $B_n(x)$ and $A_n(x)$, etc., are given in Nos. (1134) to (1146).
- (c) The coefficients $\mathcal{S}$, etc., in Nos. (305) to (312) are given in No. (1130). See also No. (1101).
- (d) The values of p_{2n+1} , q_{2n} , r_{2n+1} , and t_{2n} are given in the table opposite for values of $n = 0$ to 4.
- (e) The summation of No. (305) to 16 places of decimals is given in No. (1133).
- (f) Some of these series are derived from No. (546), etc., giving θ appropriate values.
- (g) Between Nos. (305) to (318) and (319) to (329) there is some duplication, as the series are collected from different sources, but the results are compatible.

Reference

$$= \frac{(-1)^n (2\pi)^{2n}}{2(2n+1)!} \left\{ B_{2n} \left(\frac{1}{4} \right) + (-1)^{n-1} \frac{B_n}{2n} \right\}$$

AE. 30

$$= p_{2n+1}$$

AE. 59

$$= q_{2n}$$

AE. 64

$$= r_{2n+1}$$

AE. 69

$$= t_{2n}$$

AE. 74

✓

n	p_{2n+1}	q_{2n}	r_{2n+1}	t_{2n}
0	$\frac{\pi}{2\sqrt{2}}$	—	$\frac{\pi}{3}$	—
1	$\frac{3\pi^3}{\sqrt{2} \cdot 2^6}$	$\frac{\pi^2}{\sqrt{2} \cdot 2^3}$	$\frac{7\pi^3}{6^3}$	$\frac{\pi^2}{2 \cdot 3\sqrt{3}}$
2	$\frac{19\pi^5}{\sqrt{2} \cdot 2^{12}}$	$\frac{11\pi^4}{3\sqrt{2} \cdot 2^8}$	$\frac{5 \cdot 61\pi^5}{27 \cdot 3^6}$	$\frac{23\pi^4}{2^4 \cdot 3^4\sqrt{3}}$
3	$\frac{307\pi^7}{5\sqrt{2} \cdot 2^{17}}$	$\frac{192 \cdot \pi^6}{3 \cdot 5\sqrt{2} \cdot 2^{14}}$	$\frac{61 \cdot 547\pi^7}{2^{10} \cdot 3^9 \cdot 5}$	$\frac{412\pi^6}{2^8 \cdot 3^6 \cdot 5\sqrt{3}}$
4	—	$\frac{24611\pi^8}{2^{19} \cdot 3^2 \cdot 5 \cdot 7\sqrt{2}}$	—	$\frac{11 \cdot 13 \cdot 1801\pi^8}{2^{11} \cdot 3^9 \cdot 5 \cdot 7\sqrt{3}}$

Ref.: AE. 63

AE. 66

AE. 73

AE. 77

Series No.

$$(331) \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2 \cdot (2n)!} \left\{ B_{2n+1} \left(\frac{1}{8} \right) - \frac{1}{2^{2n+1}} B_{2n+1} \left(\frac{1}{4} \right) \right\}$$

$$(332) \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n-1)!} \left\{ A_{2n} \left(\frac{1}{8} \right) - \frac{1}{2^{2n}} A_{2n} \left(\frac{1}{4} \right) \right\}$$

$$(333) \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2 \cdot (2n)!} \left\{ B_{2n+1} \left(\frac{5}{12} \right) + \frac{1}{2^{2n+1}} B_{2n+1} \left(\frac{1}{6} \right) \right. \\ \left. - \frac{1}{3^{2n+1}} B_{2n+1} \left(\frac{1}{4} \right) \right\}$$

$$(334) \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n-1)!} \left\{ A_{2n} \left(\frac{1}{12} \right) - \frac{1}{2^{2n}} A_{2n} \left(\frac{1}{6} \right) \right\}$$

In the above four series see also No. (1146) for an amplification of the coefficients A_n , B_n , etc.

$$(335) \text{ If } \sum_1^{\infty} \frac{1}{n^s}$$

$$\frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \dots \infty$$

$$\frac{1}{1^s} - \frac{1}{5^s} + \frac{1}{7^s} - \dots \infty$$

Generally

$$1 + \sum' n^{-s}$$

The dash sign indicates that only those values of n (greater than p) which are prime to $2 \cdot 3 \cdot 4 \dots p$ occur in the summation

$$(336) 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \infty$$

$$(337) 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \infty$$

$$(338) 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \dots \infty$$

Reference

$$= \frac{1}{\sqrt{2}} p_{2n+1}$$

AE. 59

$$= \frac{1}{\sqrt{2}} q_{2n}$$

AE. 64

$$= \frac{1}{2} r_{2n+1}$$

AE. 73

$$= \frac{\sqrt{3}}{2} t_{2n}$$

AE. 74

$= \zeta(s)$ 2, 3, 5... p —are prime numbers in order

$$= \zeta(s)(1 - 2^{-s})$$

$$= \zeta(s)(1 - 2^{-s})(1 - 3^{-s})$$

$$= \zeta(s)(1 - 2^{-s}) \dots (1 - p^{-s})$$

Q. 272

$$= \frac{\pi^2}{6}$$

E. 154

$$= \frac{\pi^2}{12}$$

E. 158

$$= \frac{\pi^2}{9}$$

27 P. 281



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Series No.

$$(339) 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty$$

$$(340) 1 + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \dots \infty$$

$$(341) 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots \infty$$

$$(342) 1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots \infty$$

$\Rightarrow$ Unconvergent

$$(343) 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots \infty$$

$$(344) 1 - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \dots \infty$$

$$(345) 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \dots \infty$$

$$(346) \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{31^2} + \dots \infty$$

$$(347) \frac{1}{2^2} - \frac{2}{3^2} + \frac{3}{4^2} - \frac{4}{5^2} + \dots \infty$$

$$(348) \frac{1}{1^2 \cdot 2} + \frac{1}{2^2 \cdot 3} + \frac{1}{3^2 \cdot 4} + \dots \infty$$

$$(349) \frac{1}{3^4} + \frac{1+2}{5^4} + \frac{1+2+3}{7^4} + \frac{1+2+3+4}{9^4} + \dots \infty$$

$$(350) 1^2 - \frac{2^2}{5} + \frac{3^2}{5^2} - \frac{4^2}{5^3} + \dots \infty$$

$$(351) \left(\frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} + \dots \infty \right)$$

$$+ \left(\frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \dots \infty \right)$$

Reference

$$= \frac{\pi^2}{8} \quad \text{E. 155}$$

$$= \frac{3\pi^3\sqrt{2}}{128} \quad \text{A. 364}$$

$$= \frac{\pi^3}{32} \quad \text{Y. 501}$$

$$= \frac{\pi^4}{96} \quad \text{E. 155}$$

$$= \frac{\pi^4}{90} \quad \text{E. 154}$$

$$= \frac{11\pi^4\sqrt{2}}{1536} \quad \text{A. 364}$$

$$= \frac{5\pi^5}{1536} \quad \text{Y. 501}$$

$$= \frac{\pi^2(2 - \sqrt{3})}{36} \quad \text{A. 528}$$

$$= \frac{\pi^2}{12} - \log h 2$$

$$= \frac{\pi^2 - 6}{6}$$

$$= \frac{\pi^2}{64} \left(1 - \frac{\pi^2}{12}\right) \quad \text{E. 158}$$

$$= \frac{25}{54} \quad \text{F. 332}$$

$$= \frac{\pi}{4} \quad \text{H. 462}$$

Series No.

$$(352) \quad 1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \dots \infty$$

$$(353) \quad 1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} + \dots \infty$$

$$(354) \quad 1 - \frac{1}{4^2} + \frac{1}{4^4} - \frac{1}{4^6} + \dots \infty$$

$$(355) \quad 1 - \frac{1}{5^2} + \frac{1}{5^3} - \frac{1}{5^6} + \dots \infty$$

$$(356) \quad \sum_1^n \frac{1}{n^2}$$

$$(357) \quad \sum_1^n \frac{1}{n^3}$$


$$(358) \quad 1 + \frac{1}{5^2} + \frac{1}{9^2} + \dots n \text{ terms}$$

$$(359) \quad \sum_1^\infty \frac{1}{n \cdot 2^n}$$

$$(360) \quad \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \dots \infty \text{ can be summed in five cases only:}$$

$$(a) \quad x = 1, \quad \sum_1^\infty \frac{x^n}{n^2}$$

$$(b) \quad x = -1, \quad \sum_1^\infty \frac{x^n}{n^2}$$

Bohm!



Reference

$$= \frac{4}{5} \quad \text{T. 144}$$

$$= \frac{9}{10} \quad \text{T. 144}$$

$$= \frac{16}{17} \quad \text{T. 144}$$

$$= \frac{25}{26} \quad \text{T. 144}$$

$$= \frac{\pi^2}{6} - \frac{b_1}{n+1} - \frac{b_2}{(n+1)(n+2)} - \frac{b_3}{(n+1)(n+2)(n+3)} - \dots$$

$$\text{where } b_k = \frac{(k-1)!}{k} \quad \text{T. 27}$$

$$= K - \frac{C_2}{(n+1)(n+2)} - \frac{C_3}{(n+1)(n+2)(n+3)} - \dots$$

$$\text{where } K = 1.2020569 = \sum_1^{\infty} \frac{1}{k^3}, \text{ see (1133),}$$

$$\text{and } C_k = \frac{(k-1)!}{k} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k-1} \right) \quad \text{T. 27}$$

$$= 1.0787 - \frac{1}{16} \left[\frac{1}{n+\frac{1}{4}} + \frac{1}{2(n+\frac{1}{4})} + \frac{1}{6(n+\frac{1}{4})^3} + \frac{1}{30(n+\frac{1}{4})^5} + \dots \right] \quad \text{AC. 63 (Glaisher 1876)}$$

$$= \log 2$$

$$= \frac{\pi^2}{6}$$

$$= \frac{\pi^2}{12}$$

Series No.

$$(c) \quad x = \frac{1}{2}, \quad \sum_1^{\infty} \frac{x^n}{n^2}$$

$$(d) \quad x = 2 \sin \frac{\pi}{10}, \quad \sum_1^{\infty} \frac{x^n}{n^2}$$

$$(e) \quad x = \left(2 \sin \frac{\pi}{10}\right)^2, \quad \sum_1^{\infty} \frac{x^n}{n^2}$$

$$(361) \quad \sum_1^{\infty} \frac{n^r}{n!}$$

$$(362) \quad \sum_1^{\infty} \frac{1}{n^2} \cdot \frac{1}{2^n} = \int_0^1 \log h \left(\frac{1}{x} \right) \frac{dx}{2-x}$$

$$(363) \quad \sum_1^n \left(1 + \frac{1}{x^n} \right)^2$$

$$(364) \quad \sum_1^{\infty} \frac{n}{(4n^2 - 1)^2}$$

$$(365) \quad \sum_1^{\infty} \frac{(-1)^n}{n(n+1)^2} = \int_0^1 \log h x \log h (1+x) dx$$

$$(366) \quad \sum_1^{\infty} (-1)^{n-1} \frac{1}{4n^2 - 1}$$

$$\boxed{(367)} \quad \sum_1^{\infty} \frac{1}{(2n+1)^2 - 1}$$

$$(368) \quad \sum_1^{\infty} (-1)^{n-1} \frac{1}{(2n+1)^2 - 1}$$

$$\boxed{(369)} \quad \sum_1^{\infty} \frac{1}{n^2 - 1}$$

Reference

$$= \frac{\pi^2}{12} - \frac{1}{2} \left(\log h \frac{1}{2} \right)^2$$

$$= \frac{\pi^2}{10} - \left(\log h 2 \sin \frac{\pi}{10} \right)^2$$

$$= \frac{\pi^2}{15} - \log h \left(2 \sin \frac{\pi}{10} \right)^2$$

2Z. 286

$$= \mathcal{S}_r$$

$$\mathcal{S}_1 = \epsilon \quad \mathcal{S}_2 = 2\epsilon \quad \mathcal{S}_3 = 5\epsilon \quad \mathcal{S}_4 = 15\epsilon$$

$$\mathcal{S}_5 = 52\epsilon \quad \mathcal{S}_6 = 203\epsilon \quad \mathcal{S}_7 = 877\epsilon \quad \mathcal{S}_8 = 4140\epsilon$$

A. 197

→ Nombres de Bell

$$= \frac{\pi^2}{12} - \frac{1}{2} (\log h 2)^2$$

A. 520

$$= n + \frac{2(1 - x^{-n})}{x - 1} + \frac{1 - x^{-2n}}{x^2 - 1}$$

$$= \frac{1}{8}$$

A. 52

$$= \sum (-1)^{n-1} \int_0^1 \frac{x^n}{n} \log x \, dx = 2 - 2 \log h 2 - \frac{\pi^2}{12}$$

A. 496

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$= \frac{1}{4}$$

$$= \frac{1}{2} \log h 2 - \frac{1}{4}$$

$$= \frac{3}{4}$$

Series No.

$$(370) \sum_1^{\infty} \frac{x^n}{n+a}$$

$$(371) \sum_2^n \frac{1}{n^2-1}$$

$$(372) \sum_1^{\infty} \frac{x^{n-1}}{(1-x^n)(1-x^{n+1})}$$

$$(373) \sum_1^{\infty} \frac{1}{(4n^2-1)^r}$$

$$(374) \sum_1^{\infty} \frac{1}{n(4n^2-1)}$$

$$(375) \sum_1^{\infty} \frac{1}{n(9n^2-1)}$$

$$(376) \sum_1^{\infty} \frac{1}{n(36n^2-1)}$$

$$(377) 1 + x^2 - \frac{x^3}{3} + \frac{5}{6}x^4 - \frac{3}{4}x^5 \dots \infty$$

$$(378) \sum_{n=1}^{n=\infty} \left[\frac{1}{(3n-1)^4} + \frac{1}{(3n+1)^4} \right]$$

$$(379) \frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \dots \infty$$

$$(380) 1 - \frac{n^2}{1} + \frac{n^2(n^2-1^2)}{1^2 \cdot 2^2} - \frac{n^2(n^2-1^2)(n^2-2^2)}{1^2 \cdot 2^2 \cdot 3^2}$$

... to $n+1$ terms

$$(381) \frac{x}{3} + \frac{1 \cdot 4}{3 \cdot 6} 2^3 \cdot x^2 + \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9} 3^3 \cdot x^3 + \dots \infty$$

nouveau exposité

Reference

$$= -x^{-a} \left\{ \frac{x}{1} + \frac{x^2}{2} + \dots + \frac{x^a}{a} + \log(1-x) \right\} \quad \text{where } x < 1$$

C. 246

$$= \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$$

$$= \frac{1}{(1-x)^2} \quad \text{where } |x| < 1$$

Q. 59

$$= \frac{1}{x(1-x)^2} \quad \text{where } |x| > 1$$

$$= S_r$$

$$S_1 = \frac{1}{2} \quad S_2 = \frac{\pi^2 - 8}{16} \quad S_3 = \frac{32 - 3\pi^2}{64}$$

$$S_4 = \frac{\pi^4 + 30\pi^2 - 384}{768}$$

T. 141

$$= 2 \log 2 - 1$$

T. 142

$$= \frac{3}{2} (\log 3 - 1)$$

T. 142

$$= -3 + \frac{3}{2} \log 3 + 2 \log 2$$

T. 142

$$= (1+x)^x$$

Y. 107

$$= \frac{8\pi^4}{729} - 1$$

E. 190

$$= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots \infty$$

F. 197

$$= 0$$

F. 338

$$= \frac{1}{27} \cdot \frac{x}{(1-x)^{10/3}} (x^2 + 18x + 9)$$

ref EIS

 $\approx 2/29$

Series No.

$$(382) \sum_{n=1}^{n=\infty} \frac{1}{\{(2n)^2 - (2m-1)^2\}^2}$$

$$(383) \sum_{n=1}^{n=\infty} \frac{1}{\{(2n-1)^2 - (2m)^2\}^2}$$

$$(384) \text{Lt}_{n=\infty} \frac{\left(\frac{1}{2n}\right)^p + \left(\frac{2}{2n}\right)^p + \left(\frac{3}{2n}\right)^p + \dots \text{ } 2n \text{ terms}}{\left(\frac{1}{2} + \frac{1}{2n}\right)^p + \left(\frac{1}{2} + \frac{2}{2n}\right)^p + \dots \text{ } n \text{ terms}}$$

$$(385) \sum_1^{\infty} \left\{ \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} \right\}^2 \frac{1}{2n+r}$$

$$\mathcal{S}_0 = 2 \log 2 - \frac{4}{\pi} \omega_2$$

$$\mathcal{S}_1 = \frac{4}{\pi} \omega_2 - 1$$

$$\mathcal{S}_2 = \frac{2}{\pi} - \frac{1}{2}$$

$$\mathcal{S}_3 = \frac{1}{2\pi} (2\omega_2 + 1) - \frac{1}{3}$$

$$\mathcal{S}_4 = \frac{10}{9\pi} - \frac{1}{4}$$

$$\mathcal{S}_5 = \frac{1}{32\pi} (18\omega_2 + 13) - \frac{1}{5}$$

$$\mathcal{S}_6 = \frac{178}{225\pi} - \frac{1}{6}$$

$$\mathcal{S}_7 = \frac{1}{128\pi} (50\omega_2 + 43) - \frac{1}{7}$$

$$\omega_2 = \sum_0^{\infty} (-1)^{k-1} \frac{1}{(2k+1)^2}$$

$$= 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots \infty =$$

$$= 0.9159656 \rightarrow \text{CATTALAN}$$

When r is a negative integer the value of $n = r/2$ is to be excluded in the summation.

Reference

$$= \frac{\pi^2}{16(2m-1)^2} - \frac{1}{2(2m-1)^4}$$

C. 373

$$= \frac{\pi^2}{64m^2}$$

C. 373

$$= \frac{1}{1 - \left(\frac{1}{2}\right)^{p+1}}$$

J. 74

$$= \mathcal{S}_r$$

T. 142



$$\mathcal{S}_{-1} = 1 - \frac{2}{\pi}$$

$$\mathcal{S}_{-2} = \frac{1}{2} \log h 2 + \frac{1}{4} - \frac{1}{2\pi} (2\omega_2 + 1)$$

$$\mathcal{S}_{-3} = \frac{1}{3} - \frac{10}{9\pi}$$

$$\mathcal{S}_{-4} = \frac{9}{32} \log h 2 + \frac{11}{128} - \frac{1}{32\pi} (18\omega_2 + 13)$$

$$\mathcal{S}_{-5} = \frac{1}{5} - \frac{178}{225\pi}$$

$$\mathcal{S}_{-6} = \frac{25}{128} \log h 2 + \frac{71}{1536} - \frac{1}{128\pi} (50\omega_2 + 43)$$

Series No.

$$(386) A_n = \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n}$$

$$(387) \sum_1^{\infty} A_n \frac{1}{4n^2 - 1}$$

$$(388) \sum_1^{\infty} A_n \frac{1}{2n+1}$$

$$(389) \sum_1^{\infty} (-1)^n A_n \frac{1}{2n+1}$$

$$(390) \sum_1^{\infty} A_n^2 \frac{4n+1}{(2n-1)(2n+2)}$$

$$(391) \sum_1^{\infty} (-1)^{n+1} A_n^3 \frac{4n+1}{(2n-1)(2n+2)}$$

$$(392) \sum_1^{\infty} (-1)^n A_n^3 (4n+1)$$

$$(393) \sum_1^{\infty} A_n^4 \frac{4n+1}{(2n-1)(2n+2)}$$

$$(394) \sum_{n=1}^{\infty} \frac{1}{m^2 - n^2}$$

$$(395) \sum_1^{\infty} \frac{(-1)^{n-1}}{m^2 - n^2}$$

$$(396) \sum_2^{\infty} \frac{n-1}{n!}$$

$$(397) \sum_1^{\infty} \frac{1}{4n^2 - 1}$$

$$(398) \sum_1^{\infty} \frac{12n^2 - 1}{n(4n^2 - 1)^2}$$

Reference

T. 143

$$= \frac{(2n-1)!}{2^{2n-1}n!(n-1)!}$$

$$= 1 - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - 1$$

$$= \log h(1 + \sqrt{2}) - 1$$

$$= \frac{1}{2}$$

$$= \frac{2}{\pi} - \frac{1}{2}$$

$$= \frac{2}{\pi} - 1$$

$$= \frac{1}{2} - \frac{4}{\pi^2}$$

$$= -\frac{3}{4m^2} \quad \text{where } m \text{ is an integer and } n = m \text{ omitted} \quad \text{A. 67}$$

$$= \frac{3}{4m^2} \quad \text{where } n \text{ is even and } n = m \text{ omitted} \quad \text{A. 67}$$

$$= 1 \quad \text{T. 143}$$

$$= \frac{1}{2} \quad \text{T. 141}$$

$$= 2 \log h 2$$

$$\begin{array}{r} 1 \quad 3 \quad 2 \\ 4 \quad 12 \quad 8 \\ \frac{3}{3} \quad 12 \quad 8 = \frac{1}{2} \text{ elem} \\ 3 \quad 6 \quad 4 \end{array}$$

T. 143

Series No.

$$(399) \quad 1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \dots \infty$$

$$(400) \quad \sum_2^{\infty} \frac{1}{(n^2 - 1)^2} = \left[\frac{1}{(1 \cdot 3)^2} + \frac{1}{(2 \cdot 4)^2} + \frac{1}{(3 \cdot 5)^2} + \dots \infty \right]$$

$$(401) \quad \sum_2^{\infty} \frac{(-1)^n}{(n^2 - 1)^2}$$

$$(402) \dagger \quad x + aB_2(x) + \frac{a^2}{2!} B_3(x) + \frac{a^3}{3!} B_4(x) + \dots \infty$$

$$(403) \dagger \quad 1 + a\left(x - \frac{1}{2}\right) + a^2 A_2(x) + \frac{a^3}{2!} A_3(x) + \dots \infty$$

$$(404) \quad \frac{1}{n+1} + \frac{1}{2} + \frac{n}{2!} B_1^* - \frac{n(n-1)(n-2)}{4!} B_3^* + \dots \infty$$

This series may be used to evaluate B_n^* by putting $n = 2, 4, 6$, etc.

$$(405) \quad \frac{1}{2} \left(\frac{2x}{1+x^2} \right) + \frac{1}{2 \cdot 4} \left(\frac{2x}{1+x^2} \right)^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \left(\frac{2x}{1+x^2} \right)^5 + \dots \infty$$

$$(406) \quad \sum_1^{\infty} \mathcal{S}_n \frac{x^n}{n!}$$

$$(407) \quad \sum_1^{\infty} (-1)^{n-1} \mathcal{S}_n \frac{2^n}{n!}$$

$$(408) \quad 1 + nx + \frac{n^2 x^2}{2!} + \frac{n(n^2 - 1^2)}{3!} x^3 + \frac{n^2(n^2 - 2^2)}{4!} x^4 \\ + \frac{n(n^2 - 1^2)(n^2 - 3^2)}{5!} x^5 + \dots \infty$$

$$(409) \quad 1 + \frac{1}{2^2} \left(\frac{1}{2} \right)^2 + \frac{1}{3^2} \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 + \dots \infty$$

$$(410) \quad 1 + \frac{1}{3} \cdot 1^2 + \frac{1}{5} \left(\frac{2}{3} \right)^2 + \frac{1}{7} \left(\frac{2 \cdot 4}{3 \cdot 5} \right)^2 + \dots \infty$$

† For values of $A_n(x)$ and $B_n(x)$, see No. (1146).

Reference

$$= \frac{4}{5}$$

T. 144

$$= \frac{\pi^2}{12} - \frac{11}{16}$$

$$= \frac{\pi^2}{24} - \frac{5}{16}$$

$$= \frac{e^{ax} - 1}{e^a - 1} \quad \text{where } x \text{ is a positive integer}$$

AE. 6

$$= \frac{ae^{ax}}{e^a - 1}$$

AE. 20

$$= 1$$

Y. 109

$$= x \quad \text{where } 1 > x > -1$$

Y. 459

$$= \frac{1}{x} \quad \text{where } x > 1$$

$$= \frac{1}{4} x e^x (x^3 + 8x^2 + 14x + 4) \quad \text{where } \mathcal{S}_n = 1^3 + 2^3 + 3^3 + \dots n^3$$

A. 197

$$= 0 \quad \text{where } \mathcal{S}_n = 1^3 + 2^3 + 3^3 + \dots n^3$$

A. 197

$$= (x + \sqrt{1 + x^2})^n$$

Y. 107

$$= \frac{11}{\pi} - 4$$

1Z. 360

$$= \frac{\pi}{2}$$

1Z. 360

Series No.

$$(411) \frac{1}{3^2} \cdot 1^2 + \frac{1}{5^2} \left(\frac{2}{3}\right)^2 + \frac{1}{7^2} \left(\frac{2 \cdot 4}{3 \cdot 5}\right)^2 + \dots \infty$$

$$(412) \frac{x(a+x)}{a} - \frac{x^2(a+x)^2}{a^3} + \frac{4}{1 \cdot 2} \frac{x^3(a+x)^3}{a^5} + \dots$$

$$+ (-1)^{n-1} \frac{(2n-2)!}{n!(n-1)!} \frac{x^n(a+x)^n}{a^{2n-1}} + \dots \infty$$

$$(413) 1 - ax(1-x) + \frac{a(a-3)}{2!} x^2(1-x)^2$$

$$- \frac{a(a-4)(a-5)}{3!} x^3(1-x)^3 + \dots$$

$$(414) \lim_{x \rightarrow 1} \sum_1^{\infty} (-1)^{n-1} \frac{x^n}{n(1+x^n)}$$

$$(415) \lim_{x \rightarrow 1} \sum_1^{\infty} \frac{x^n}{1-x^n}$$

$$(416) \lim_{x \rightarrow 1} \left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \dots \infty \right)$$

XIV. Trigonometrical Summations

$$(417) \sin \theta + \sin 2\theta + \sin 3\theta + \text{to } n \text{ terms}$$

$$(418) \cos \theta + \cos 2\theta + \cos 3\theta + \text{to } n \text{ terms}$$

$$(419) \sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin (2n-1)\theta$$

$$(420) \cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos (2n-1)\theta$$

$$(421) \cos \theta + \sin 3\theta + \cos 5\theta + \dots + \sin (4n-1)\theta$$

Reference

$$= \pi - 3$$

1Z. 360

$$= x$$

A. 199

$$= (1 - x)^a \quad \text{where} \quad |x(1 - x)| < \frac{1}{4}$$

A. 199

$$= \frac{1}{2} \log 2$$

A. 201

$$\approx \frac{1}{1 - x} \log \frac{1}{1 - x}$$

A. 201

$$\approx \frac{\pi}{4} \cdot \frac{1}{1 - x}$$

A. 201

$$= \sin \frac{1}{2} (n + 1) \theta \sin \frac{n\theta}{2} \operatorname{cosec} \frac{\theta}{2}$$

E. 283

$$= \cos \frac{1}{2} (n + 1) \theta \sin \frac{n\theta}{2} \operatorname{cosec} \frac{\theta}{2}$$

E. 283

$$= \sin^2 n\theta \cdot \operatorname{cosec} \theta$$

E. 283

$$= \frac{1}{2} \sin 2n\theta \cdot \operatorname{cosec} \theta$$

E. 287

$$= \sin 2n\theta \{ \cos 2n\theta + \sin 2n\theta \} \times \{ \cos \theta + \sin \theta \} \operatorname{cosec} 2\theta$$

E. 288

Series No.

$$(422) \operatorname{cosec} \theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta + \dots n \text{ terms}$$

$$(423) \cos \frac{\theta}{2} + \cos 2\theta + \cos \frac{7\theta}{2} + \dots n \text{ terms}$$

$$(424) \cos \frac{\pi}{2n+1} + \cos \frac{3\pi}{2n+1} + \dots n \text{ terms}$$

$$(425) 1 - 2 \cos \theta + 3 \cos 2\theta - 4 \cos 3\theta + \dots n \text{ terms}$$

$$(426) 3 \sin \theta + 5 \sin 2\theta + 7 \sin 3\theta + \dots n \text{ terms}$$

$$(427) \sum_1^{n-1} k \sin k\theta$$

$$(428) \sum_1^{n-1} k \cos k\theta$$

$$(429) \sum_1^n (-1)^k \cos k\theta$$

$$(430) \sum_1^n \sin (2k-1)\theta$$

$$(431) \sum_1^{n+1} (-1)^{k-1} \sin (2k-1)\theta$$

$$(432) \tan \theta + \tan \left(\theta + \frac{\pi}{n} \right) + \tan \left(\theta + \frac{2\pi}{n} \right) + \dots n \text{ terms}$$

Reference

$$= \cot \frac{\theta}{2} - \cot 2^{n-1}\theta \quad \text{E. 125}$$

$$= \cos \frac{1}{4} (3n - 1)\theta \sin \frac{3n\theta}{4} \operatorname{cosec} \frac{3\theta}{4} \quad \text{E. 287}$$

$$= \frac{1}{2} \quad \text{E. 288}$$

$$= \frac{\cos \theta + (-1)^{n-1}\{(n+1) \cos (n-1)\theta + n \cos n\theta\}}{2(1 + \cos \theta)} \quad \text{E. 117}$$

$$= \frac{\sin \theta + (2n+3) \sin n\theta - (2n+1) \sin (n+1)\theta}{2(1 - \cos \theta)} \quad \text{E. 117}$$

$$= \frac{\sin n\theta}{4 \sin^2 \frac{\theta}{2}} - \frac{n \cos \left(\frac{2n-1}{2}\theta\right)}{2 \sin \frac{\theta}{2}} \quad \text{T. 82}$$

$$= \frac{n \sin \left(\frac{2n-1}{2}\theta\right)}{2 \sin \frac{\theta}{2}} - \frac{1 - \cos n\theta}{4 \sin^2 \frac{\theta}{2}} \quad \text{T. 82}$$

$$= -\frac{1}{2} + (-1)^n \frac{\cos \left(\frac{2n+1}{2}\theta\right)}{2 \cos \frac{\theta}{2}} \quad \text{T. 82}$$

$$= \frac{\sin^2 n\theta}{\sin \theta} \quad \text{T. 82}$$

$$= (-1)^n \frac{\sin (2n+2)\theta}{2 \cos \theta} \quad \text{T. 82}$$

$$= -n \cot \left(\frac{n\pi}{2} + n\theta\right) \quad \text{E. 73}$$

Series No.

$$(433) \cot \theta + \cot \left(\theta + \frac{\pi}{n} \right) + \cot \left(\theta + \frac{2\pi}{n} \right) + \dots n \text{ terms}$$

$$(434) \sum_0^{n-1} \cos \frac{2\pi k^2}{n}$$

$$(435) \sum_1^{n-1} \sin \frac{2\pi k^2}{n}$$

$$(436) \sum_1^{n-1} \sin \frac{\pi k}{n}$$

$$(437) \sum_1^n \sin^2 k\theta$$

$$(438) \sum_0^n \cos^2 k\theta$$

$$(439) \frac{1}{\sin^2 \theta} + \frac{1}{\sin^2 2\theta} + \dots + \frac{1}{\sin^2 \frac{n-1}{2} \theta}$$

$$(440) \frac{1}{\sin^2 \theta} + \frac{1}{\sin^2 2\theta} + \dots + \frac{1}{\sin^2 \frac{n-2}{2} \theta}$$

$$(441) \frac{1}{\sin^2 \theta} + \frac{1}{\sin^2 3\theta} + \dots + \frac{1}{\sin^2 (n-2)\theta}$$

$$(442) \frac{1}{\sin^2 \theta} + \frac{1}{\sin^2 3\theta} + \dots + \frac{1}{\sin^2 (n-1)\theta}$$

$$(443) \sum_1^{n-1} \operatorname{cosec}^2 \left(\frac{r\pi}{n} \right)$$

$$(444) \operatorname{cosec}^2 \theta + \operatorname{cosec}^2 \left(\theta + \frac{2\pi}{n} \right) + \dots n \text{ terms}$$

$$(445) \tan^2 \theta + \tan^2 \left(\theta + \frac{\pi}{n} \right) + \dots n \text{ terms}$$

Reference

$$= n \cot n\theta \quad \text{E. 73}$$

$$= \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right) \quad \text{T. 83}$$

$$= \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} \right) \quad \text{T. 83}$$

$$= \cot \frac{\pi}{2n} \quad \text{T. 83}$$

$$= \frac{n}{2} - \frac{\cos(n+1)\theta \sin n\theta}{2 \sin \theta} \quad \text{T. 82}$$

$$= \frac{n+2}{2} + \frac{\cos(n+1)\theta \sin n\theta}{2 \sin \theta} \quad \text{T. 82}$$

$$= \frac{n^2 - 1}{6} \quad \text{where } n \text{ is odd} \quad \text{A. 210}$$

$$= \frac{n^2 - 4}{6} \quad \text{where } n \text{ is even} \quad \text{A. 210}$$

$$= \frac{n^2 - 1}{2} \quad \text{where } n \text{ is odd} \quad \text{A. 211}$$

$$= \frac{n^2}{2} \quad \text{where } n \text{ is even} \quad \text{A. 211}$$

$$= \frac{1}{3}(n^2 - 1) \quad \text{where } n \text{ is odd} \quad \text{A. 223}$$

$$= n^2 \operatorname{cosec}^2 n\theta \quad \text{where } n \text{ is odd}$$

$$= \frac{1}{2} n^2 \operatorname{cosec}^2 \frac{n\theta}{2} \quad \text{where } n \text{ is even} \quad \text{E. 73}$$

$$= n^2 \cot^2 \left(\frac{n\pi}{2} + n\theta \right) + n(n-1) \quad \text{E. 73}$$

Series No.

$$(446) \sin^3 \theta + \sin^3 2\theta + \sin^3 3\theta + \dots n \text{ terms}$$

$$(447) \cos^3 \theta + \cos^3 2\theta + \cos^3 3\theta + \dots n \text{ terms}$$

$$(448) \sin^4 \theta + \sin^4 2\theta + \sin^4 3\theta + \dots n \text{ terms}$$

$$(449) \cos^4 \theta + \cos^4 2\theta + \cos^4 3\theta + \dots n \text{ terms}$$

$$(450) \sum_1^{\frac{n-1}{2}} \tan^4 \left(\frac{r\pi}{n} \right)$$

$$(451) \cot^2 \frac{2\pi}{2n} + \cot^2 \frac{4\pi}{2n} + \dots + \cot^2 \frac{(n-1)\pi}{2n}$$

$$(452) \cot^4 \frac{2\pi}{2n} + \cot^4 \frac{4\pi}{2n} + \dots + \cot^4 \frac{(n-1)\pi}{2n}$$

$$(453) 1 + a \cos \theta + a^2 \cos 2\theta + \dots + a^{n-1} \cos (n-1)\theta$$

$$(454) \sum_0^{n-1} a^k \cos k\theta$$

Nos. (453) and (454) are equal.

$$(455) \sum_1^{n-1} a^k \sin k\theta$$

Reference

$$= \frac{3}{4} \sin \frac{1}{2}(n+1)\theta \sin \frac{n\theta}{2} \operatorname{cosec} \frac{\theta}{2} - \frac{1}{4} \sin \frac{3}{2}(n+1)\theta \sin \frac{3n\theta}{2} \operatorname{cosec} \frac{3\theta}{2}$$

E. 288

$$= \frac{3}{4} \cos \frac{1}{2}(n+1)\theta \sin \frac{n\theta}{2} \operatorname{cosec} \frac{\theta}{2} + \frac{1}{4} \cos \frac{3}{2}(n+1)\theta \sin \frac{3n\theta}{2} \operatorname{cosec} \frac{3\theta}{2}$$

E. 285

$$= \frac{1}{8} [3n - 4 \cos(n+1)\theta \sin n\theta \operatorname{cosec} \theta + \cos 2(n+1)\theta \sin 2n\theta \operatorname{cosec} 2\theta]$$

E. 288

$$= \frac{1}{8} [3n + 4 \cos(n+1)\theta \sin n\theta \operatorname{cosec} \theta + \cos 2(n+1)\theta \sin 2n\theta \operatorname{cosec} 2\theta]$$

E. 288

$$= \frac{1}{6} n(n-1)(n^2+n-3) \quad \text{where } n \text{ is odd}$$

A. 223

$$= \frac{1}{6} (n-1)(n-2) \quad \text{where } n \text{ is odd}$$

O. 349

$$= \frac{1}{90} (n-1)(n-2)(n^2+3n-13) \quad \text{where } n \text{ is odd}$$

O. 349

$$= \frac{1 - a \cos \theta + a^{n+1} \cos(n-1)\theta - a^n \cos n\theta}{1 - 2a \cos \theta + a^2}$$

$$= \frac{(1 - a \cos \theta)(1 - a^n \cos n\theta) + a^{n+1} \sin \theta \sin n\theta}{1 - 2a \cos \theta + a^2}$$

T. 82

$$= \frac{a \sin \theta (1 - a^n \cos n\theta) - (1 - a \cos \theta) a^n \sin n\theta}{1 - 2a \cos \theta + a^2}$$

T. 82

Series No.

$$(456) a \sin \theta + 2a^2 \sin 2\theta + 3a^3 \sin 3\theta + \dots n \text{ terms}$$

$$(457) \cos^3 \theta - \frac{1}{3} \cos^3 3\theta + \frac{1}{3^2} \cos^3 3^2\theta + \frac{1}{3^3} \cos^3 3^3\theta + \dots \text{ to } n \text{ terms}$$

$$(458) \cos \frac{\pi}{n} + \cos \frac{2\pi}{n} + \dots + \cos \frac{(2n-1)\pi}{n}$$

$$(459) \sum_0^n \frac{1}{2^n} \tan \frac{\theta}{2^n}$$

$$(460) \sum_1^n \left(\frac{1}{2^n} \sec \frac{\theta}{2^n} \right)^2$$

$$(461) \sum_1^n \left(2^n \sin^2 \frac{\theta}{2^n} \right)^2$$

$$(462) \sum_0^n \left(\frac{1}{2^{2n}} \tan^2 \frac{\theta}{2^n} \right)$$

$$(463) \sum_1^{\frac{n-1}{2}} \frac{1}{n^2 \sin^2 \left(\frac{r\pi}{n} \right)}$$

$$(464) \sum_1^{\frac{n-1}{2}} \frac{1}{n^2 \sin^2 \left(\frac{r\pi}{2n} \right)}$$

$$(465) \sin \theta + \sin (\theta + \beta) + \sin (\theta + 2\beta) + \dots n \text{ terms}$$

$$(466) \cos \theta + \cos (\theta + \beta) + \cos (\theta + 2\beta) + \dots n \text{ terms}$$

$$(467) \sin \theta - \sin (\theta + \beta) + \sin (\theta + 2\beta) - \dots n \text{ terms}$$

$$(468) \cos \theta - \cos (\theta + \beta) + \cos (\theta + 2\beta) - \dots 2n \text{ terms}$$

Reference

$$\begin{aligned}
 &= [a \sin \theta - a^3 \sin \theta - (n+1)a^{n+1} \sin (n+1)\theta \\
 &\quad + 2(n+1)a^{n+2} \sin n\theta - (n+1)a^{n+3} \sin (n-1)\theta \\
 &\quad + na^{n+2} \sin (n+2)\theta - 2na^{n+3} \sin (n+1)\theta \\
 &\quad + na^{n+4} \sin n\theta] / (1 - 2a \cos \theta + a^2)^2
 \end{aligned}$$

D. 502

$$= \frac{1}{4} \left\{ 3 \cos \theta + \left(-\frac{1}{3} \right)^{n-1} \cos 3n\theta \right\}$$

E. 126

$$= -1$$

$$= \frac{1}{2^n} \cot \frac{\theta}{2^n} - 2 \cot 2\theta$$

T. 83

$$= \operatorname{cosec}^2 \theta - \left(\frac{1}{2^n} \operatorname{cosec} \frac{\theta}{2^n} \right)^2$$

T. 82

$$= \left(2^n \sin \frac{\theta}{2^n} \right)^2 - \sin^2 \theta$$

T. 82

$$= \frac{2^{2n+2} - 1}{3 \cdot 2^{2n-1}} + 4 \cot^2 2\theta - \frac{1}{2^{2n}} \cot \frac{\theta}{2^n}$$

T. 83

$$= \frac{n^2 - 1}{6n^2} \quad \text{where } n \text{ is odd}$$

A. 218

$$= \frac{1}{2} \quad \text{where } r \text{ is odd and } n \text{ is even}$$

A. 218

$$= \sin \left\{ \theta + \frac{1}{2} (n-1)\beta \right\} \sin \frac{n\beta}{2} \operatorname{cosec} \frac{\beta}{2}$$

E. 282

$$= \cos \left\{ \theta + \frac{1}{2} (n-1)\beta \right\} \sin \frac{n\beta}{2} \operatorname{cosec} \frac{\beta}{2}$$

E. 283

$$= \sin \left\{ \theta + \frac{n-1}{2} (\beta + \pi) \right\} \sin \frac{n(\beta + \pi)}{2} \sec \frac{\beta}{2}$$

E. 285

$$= \sin \left\{ \theta + \left(n - \frac{1}{2} \right) \beta \right\} \sin n\beta \sec \frac{\beta}{2}$$

E. 288

Series No.

$$(469) \sin \theta \cdot \sin 2\theta + \sin 2\theta \cdot \sin 3\theta + \dots n \text{ terms}$$

$$(470) \cos \theta \cdot \sin 2\theta + \sin 2\theta \cdot \cos 3\theta + \dots 2n \text{ terms}$$

$$(471) \sin \theta \cdot \sin 3\theta + \sin 2\theta \cdot \sin 4\theta + \dots n \text{ terms}$$

$$(472) \cos \theta \sin \beta + \cos 3\theta \sin 2\beta + \cos 5\theta \sin 3\beta + \dots n \text{ terms}$$

$$(473) \sum_1^n r \sin (\phi + r\theta)$$

$$(474) -6 \sin (a + b) - 5 \sin (2a + b) - \dots \\ + (n - 7) \sin (na + b)$$

$$(475)^\dagger \sin \theta + a \sin (\theta + \beta) + a^2 \sin (\theta + 2\beta) + \dots n \text{ terms}$$

$$(476) \tan \theta \tan (\theta + \beta) + \tan (\theta + \beta) \tan (\theta + 2\beta) + \dots n \text{ terms}$$

$$(477) \operatorname{cosec} \theta \operatorname{cosec} 2\theta + \operatorname{cosec} 2\theta \operatorname{cosec} 3\theta + \dots n \text{ terms}$$

$$(478) \sec \theta \sec 2\theta + \sec 2\theta \sec 3\theta + \dots n \text{ terms}$$

$$(479) \sum_0^n \sin (\theta + k\beta)$$

$$^\dagger \text{ Sum to infinity} = \frac{\sin \theta - a \sin (\theta - \beta)}{1 - 2a \cos \beta + a^2} \quad \text{where } a < 1.$$

Reference

$$= \frac{1}{4} \{(n+1) \sin 2\theta - \sin 2(n+1)\theta\} \operatorname{cosec} \theta \quad \text{E. 288}$$

$$= \frac{1}{2} \sin 2(n+1)\theta \cdot \sin 2n\theta \operatorname{cosec} \theta \quad \text{E. 288}$$

$$= \frac{n}{2} \cos 2\theta - \frac{1}{2} \cos (n+3)\theta \cdot \sin n\theta \cdot \operatorname{cosec} \theta \quad \text{E. 288}$$

$$\begin{aligned} &= \frac{1}{2} \sin \left\{ n\theta + \frac{1}{2}(n+1)\beta \right\} \sin \frac{n}{2}(2\theta + \beta) \operatorname{cosec} \frac{1}{2}(2\theta + \beta) \\ &\quad - \frac{1}{2} \sin \left\{ n\theta - \frac{1}{2}(n+1)\beta \right\} \sin \frac{n}{2}(2\theta - \beta) \operatorname{cosec} \frac{1}{2}(2\theta - \beta) \end{aligned} \quad \text{E. 286}$$

$$\begin{aligned} &= \frac{(n+1) \sin (\phi + n\theta) - \sin \phi - n \sin (\phi + n+1\theta)}{2(1 - \cos \theta)} \\ &= (n-7) \sin (na+b) + (n-8) \frac{\sin \left(na+b - \frac{a+\pi}{2} \right)}{2 \sin a/2} \\ &\quad - \frac{\sin \{na+b - (a+\pi)\}}{2(\sin a/2)^2} - \frac{7 \sin (a+b) - 6 \sin b}{(2 \sin a/2)^2} \\ &= \frac{\sin \theta - a \sin (\theta - \beta) - a^n \sin (\theta + n\beta) + a^{n+1} \sin \{\theta + (n-1)\beta\}}{1 - 2a \cos \beta + a^2} \end{aligned} \quad \text{E. 117}$$

$$= \frac{\tan (\theta + n\beta) - \tan \theta - n \tan \beta}{\tan \beta} \quad \text{E. 124}$$

$$= \operatorname{cosec} \theta \{\cot \theta - \cot (n+1)\theta\} \quad \text{E. 125}$$

$$= \operatorname{cosec} \theta \{\tan (n+1)\theta - \tan \theta\} \quad \text{E. 125}$$

$$= \sin \left(\theta + \frac{n\beta}{2} \right) \sin \left(\frac{n+1}{2} \right) \beta \operatorname{cosec} \frac{\beta}{2} \quad \text{T. 82}$$

Series No.

$$(480) \sum_0^n \cos(\theta + k\beta)$$

$$(481) 1 - \frac{n^2 - 1^2}{2!} \sin^2 \theta + \frac{(n^2 - 1^2)(n^2 - 3^2)}{4!} \sin^4 \theta + \dots$$

$$+ (-1)^{(n-1)/2} 2^{n-1} \sin^{n-1} \theta$$

$$(482) n \sin \theta - \frac{n(n^2 - 1^2)}{3!} \sin^3 \theta + \frac{n(n^2 - 1^2)(n^2 - 3^2)}{5!} \sin^5 \theta + \dots$$

$$+ (-1)^{(n-1)/2} 2^{n-1} \sin^n \theta$$

$$(482a) 1 - \frac{n^2 - 1^2}{2!} \cos^2 \theta + \frac{(n^2 - 1^2)(n^2 - 3^2)}{4!} \cos^4 \theta$$

$$- \frac{(n^2 - 1^2)(n^2 - 3^2)(n^2 - 5^2)}{6!} \cos^6 \theta + \dots$$

$$+ (-1)^{(n-1)/2} (2 \cos \theta)^{n-1}$$

$$(482b) n \cos \theta - \frac{n(n^2 - 2^2)}{3!} \cos^3 \theta$$

$$+ \frac{n(n^2 - 2^2)(n^2 - 4^2)}{5!} \cos^5 \theta - \dots$$

$$+ (-1)^{n/2+1} (2 \cos \theta)^{n-1}$$

$$(482c) n \cos \theta - \frac{n(n^2 - 1^2)}{3!} \cos^3 \theta$$

$$+ \frac{n(n^2 - 1^2)(n^2 - 3^2)}{5!} \cos^5 \theta - \dots$$

$$+ (-1)^{(n-1)/2} 2^{n-1} \cos^n \theta$$

$$(482d) 1 - \frac{n^2}{2!} \cos^2 \theta + \frac{n^2(n^2 - 2^2)}{4!} \cos^4 \theta$$

$$- \frac{n^2(n^2 - 2^2)(n^2 - 4^2)}{6!} \cos^6 \theta + \dots + (-1)^{n/2} 2^{n-1} \cos^n \theta$$

Reference

$$= \cos \left(\theta + \frac{n\beta}{2} \right) \sin \left(\frac{n+1}{2} \right) \beta \operatorname{cosec} \frac{\beta}{2}$$

T. 82

$$= \frac{\cos n\theta}{\cos n} \quad \text{where } n \text{ is odd}$$

A. 204

$$= \sin n\theta \quad \text{where } n \text{ is odd}$$

A. 205

$$= (-1)^{(n-1)/2} \frac{\sin n\theta}{\sin \theta} \quad \text{where } n \text{ is odd}$$

$$\begin{cases} \text{E. 64} \\ \text{A. 204} \end{cases}$$

$$= (-1)^{n/2+1} \frac{\sin n\theta}{\sin \theta} \quad \text{where } n \text{ is even}$$

$$\begin{cases} \text{E. 65} \\ \text{A. 204} \end{cases}$$

$$= (-1)^{(n-1)/2} \cos n\theta \quad \text{where } n \text{ is odd}$$

$$\begin{cases} \text{E. 67} \\ \text{A. 204} \end{cases}$$

$$= (-1)^{n/2} \cos n\theta \quad \text{where } n \text{ is even}$$

$$\begin{cases} \text{E. 68} \\ \text{A. 204} \end{cases}$$

Series No.

$$(483) \quad n \sin \theta - \frac{n(n^2 - 2^2)}{3!} \sin^3 \theta \\ + \frac{n(n^2 - 2^2)(n^2 - 4^2)}{5!} \sin^5 \theta + \dots \\ + (-1)^{n/2+1} (2 \sin \theta)^{n-1}$$

$$(484)^\dagger \quad 1 - \frac{n^2}{2!} \sin^2 \theta + \frac{n^2(n^2 - 2^2)}{4!} \sin^4 \theta + \dots \\ + (-1)^{n/2} 2^{n-1} \sin^n \theta$$

$$(485) \quad \sum_1^{\frac{n-1}{2}} \{ \cot(\theta + ra) + \cot(\theta - ra) \}$$

$$(486) \quad \sum_1^{\frac{n-1}{2}} \{ \operatorname{cosec}^2(\theta + ra) + \operatorname{cosec}^2(\theta - ra) \}$$

$$(487) \quad \frac{\sin \theta}{2 \cos \theta - 1} + \frac{2 \sin 2\theta}{2 \cos 2\theta - 1} + \frac{2^2 \sin 2^2\theta}{2 \cos 2^2\theta - 1} + \dots \\ n \text{ terms}$$

$$(488) \quad \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{13} + \tan^{-1} \frac{1}{21} + \dots \\ + \tan^{-1} \frac{1}{1 + n(n+1)}$$

$$(489) \quad 2 \sum_1^{\frac{n-2}{2}} \frac{x - a \cos \frac{2r\pi}{n}}{x^2 - 2ax \cos \frac{2r\pi}{n} + a^2}$$

$$(490) \quad 2 \sum_1^{\frac{n-1}{2}} \frac{x - a \cos \frac{2r\pi}{n}}{x^2 - 2ax \cos \frac{2r\pi}{n} + a^2}$$

† This summation can be carried to infinity, the sum being $\cos \frac{1}{2} n\pi$, where the value of n is unrestricted, and $\theta = \frac{\pi}{2}$.

$$= \frac{\sin n\theta}{\cos \theta} \quad \text{where } n \text{ is even} \quad \text{A. 205}$$

$$= \cos n\theta \quad \text{where } n \text{ is even} \quad \text{A. 204}$$

$$= n \cot n\theta - \cot \theta \quad \text{where } na = \pi \quad \text{A. 217}$$

$$= n^2 \operatorname{cosec}^2 n\theta - \operatorname{cosec}^2 \theta \quad \text{where } na = \pi \quad \text{A. 217}$$

$$= \frac{2^n \sin 2^n \theta}{2^n \cos 2^n \theta + 1} - \frac{\sin \theta}{2 \cos \theta + 1} \quad \text{D. 330}$$

$$= \tan^{-1} \frac{n}{n+2} \quad \text{E. 126}$$

$$= \frac{nx^{n-1}}{x^n - a^n} - \frac{1}{x-a} - \frac{1}{x+a} \quad \text{where } n \text{ is even} \quad \text{Y. 55}$$

$$= \frac{nx^{n-1}}{x^n - a^n} - \frac{1}{x-a} \quad \text{where } n \text{ is odd} \quad \text{Y. 55}$$

$$\text{A. 207}$$

Series No.

$$(491) \quad \lim_{n \rightarrow \infty} \left[\frac{1}{n} \left\{ \sec \frac{\theta}{n} + \sec \frac{2\theta}{n} + \dots + \sec \frac{(n-1)\theta}{n} \right\} \right]$$

$$(492) \quad \sum_1^n \tan^{-1} \left(\sec \frac{2r\pi}{2n+1} \sinh \theta \right)$$

$$(493) \quad (2 \cos \theta)^{n-1} - (n-2)(2 \cos \theta)^{n-3} \\ + \frac{(n-3)(n-4)}{2!} (2 \cos \theta)^{n-5} + \\ \dots + (-1)^{(n-1)/2} \\ \dots + (-1)^{n/2-1} (n \cos \theta)$$

$$(494) \quad (2 \cos \theta)^n - n(2 \cos \theta)^{n-2} + \frac{n(n-3)}{2!} (2 \cos \theta)^{n-4} + \\ \dots + (-1)^{(n-1)/2} 2n \cos \theta \quad (n \text{ is odd}) \\ \dots + 2(-1)^{n/2} \quad (n \text{ is even})$$

$$(495) \quad \frac{1}{\cos \theta + \cos 3\theta} + \frac{1}{\cos 3\theta + \cos 5\theta} + \dots \quad n \text{ terms}$$

$$(496) \quad \lim_{n \rightarrow \infty} \left[\frac{1}{n} \left\{ \sin^{2k} \frac{\pi}{2n} + \sin^{2k} \frac{2\pi}{2n} + \dots + \sin^{2k} \frac{\pi}{2} \right\} \right]$$

$$(497) \quad \lim_{x \rightarrow 1} [1 - x \cos \theta + x^4 \cos 2\theta - x^9 \cos 3\theta + \dots \infty]$$

$$(498) \quad \lim_{x \rightarrow 1} [x \sin \theta - x^4 \sin 2\theta + x^9 \sin 3\theta - \dots \infty]$$

$$(499) \quad \sum_1^\infty a^n \sin n\theta$$

$$(500) \quad \sum_0^\infty a^n \cos n\theta$$

$$(501) \quad \cos \theta + a \cos 3\theta + a^2 \cos 5\theta + \dots a^n \cos (2n+1)\theta + \\ \dots \infty$$

Reference

$$= \frac{1}{\theta} \log \tan \left(\frac{\theta}{2} + \frac{\pi}{4} \right) \quad \text{where } \pi > \theta > \frac{\pi}{2}$$

1Z. 355

$$= \tan^{-1} \frac{\sinh n\theta}{\cosh (n+1)\theta} \quad \text{where } n \text{ is even}$$

A. 528

$$= \sin n\theta \cdot \operatorname{cosec} \theta \quad \text{where } n \text{ is odd}$$

E. 61

$$= \sin n\theta \cdot \operatorname{cosec} \theta \quad \text{where } n \text{ is even}$$

$$= 2 \cos n\theta$$

E. 63

$$= \frac{1}{2} \operatorname{cosec} \theta \{ \tan (n+1)\theta - \tan \theta \}$$

E. 125

$$= \frac{(2k-1)(2k-3)\dots 1}{2k(2k-2)\dots 2} \quad \text{where } k \text{ is a positive integer}$$

1Z. 326

$$= \frac{1}{2}$$

A. 276

$$= \frac{1}{2} \tan \frac{\theta}{2}$$

A. 276

$$= \frac{a \sin \theta}{1 - 2a \cos \theta + a^2} \quad \text{where } a^2 < 1$$

T. 139

$$= \frac{1 - a \cos \theta}{1 - 2a \cos \theta + a^2} \quad \text{where } a^2 < 1$$

T. 139

$$= \frac{(1-a) \cos \theta}{1 - 2a \cos 2\theta + a^2} \quad \text{where } |a| < 1$$

A. 223

Series No.

$$(502) \tan \theta + \tan \left(\theta + \frac{2\pi}{5} \right) + \tan \left(\theta + \frac{4\pi}{5} \right) + \dots \tan \left(\theta + \frac{8\pi}{5} \right)$$

$$(503) \cos \theta + \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta + \dots \infty$$

$$(504) \cos \theta - \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta + \dots \infty$$

$$(505) \cos \theta + \frac{1}{3} \cos 3\theta + \frac{1}{5} \cos 5\theta + \dots \infty$$

$$(506) \cos \theta - \frac{1}{3} \cos 3\theta + \frac{1}{5} \cos 5\theta + \dots \infty$$

$$(507) \cos 2\theta + \frac{1}{2} \cos 4\theta + \frac{1}{3} \cos 6\theta + \dots \infty$$

$$(508) \sin \theta + \frac{1}{2} \sin 2\theta + \frac{1}{3} \sin 3\theta + \dots \infty$$

$$(509) \sin \theta - \frac{1}{2} \sin 2\theta + \frac{1}{3} \sin 3\theta + \dots \infty$$

$$(510) \sin \theta + \frac{1}{3} \sin 3\theta + \frac{1}{5} \sin 5\theta + \dots \infty$$

$$(511) \sin \theta - \frac{1}{3} \sin 3\theta + \frac{1}{5} \sin 5\theta - \dots \infty$$

$$(512) \sin 2\theta + \frac{1}{2} \sin 4\theta + \frac{1}{3} \sin 6\theta + \dots \infty$$

Reference

$$= 5 \tan 5\theta \quad \text{E. 191}$$

$$= -\log h 2 \sin \frac{\theta}{2} \quad \text{where } 0 < \theta < 2\pi \quad \text{A. 356}$$

$$= \log h 2 \cos \frac{\theta}{2} \quad \text{where } -\pi < \theta < \pi \quad \text{A. 356}$$

$$= \frac{1}{2} \log h \cot \frac{\theta}{2} \quad \text{where } 0 < \theta < \pi \quad \text{A. 356}$$

$$= \frac{1}{2} \log h \left(-\cot \frac{\theta}{2} \right) \quad \text{where } \pi < \theta < 2\pi \quad \text{A. 356}$$

$$= \frac{\pi}{4} \quad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \text{A. 359}$$

$$= -\log h (2 \sin \theta) \quad \text{where } 0 < \theta < \pi \quad \text{A. 356}$$

$$= \frac{1}{2} (\pi - \theta) \quad \text{where } 0 < \theta < 2\pi \quad \text{A. 356}$$

$$= \frac{\theta}{2} \quad \text{where } -\pi < \theta < \pi \quad \text{A. 356}$$

$$= \frac{\pi}{4} \quad \text{where } 0 < \theta < \pi \quad \text{A. 356}$$

$$= -\frac{\pi}{4} \quad \text{where } \pi < \theta < 2\pi \quad \text{A. 356}$$

$$= \frac{1}{2} \log h (\sec \theta + \tan \theta) \quad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \text{A. 359}$$

$$= \frac{1}{2} (\pi - 2\theta) \quad \text{where } 0 < \theta < \pi \quad \text{A. 356}$$

Series No.

$$(513) \cos \frac{\pi}{3} + \frac{1}{3} \cos \frac{2\pi}{3} + \frac{1}{5} \cos \frac{3\pi}{3} + \frac{1}{7} \cos \frac{4\pi}{3} + \dots \infty$$

$$(514) \tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta - \dots \infty$$

$$(515) \frac{1}{2} \sin \theta - \frac{2}{5} \sin 2\theta + \frac{3}{10} \sin 3\theta - \frac{4}{17} \sin 4\theta + \dots \infty$$

$$(516) \cos \theta + \frac{1}{2^2} \cos 2\theta + \frac{1}{3^2} \cos 3\theta + \dots \infty$$

$$(517) \cos \theta - \frac{1}{2^2} \cos 2\theta + \frac{1}{3^2} \cos 3\theta - \dots \infty$$

$$(518) \cos \theta + \frac{1}{3^2} \cos 3\theta + \frac{1}{5^2} \cos 5\theta + \dots \infty$$

$$(519) \sin \theta - \frac{1}{3^2} \sin 3\theta + \frac{1}{5^2} \sin 5\theta - \dots \infty$$

$$(520) \sum_1^{\infty} \frac{\sin^2 n\theta}{n^2}$$

$$(521) \sin \theta + \frac{1}{2} \sin 2\theta + \frac{1}{2^2} \sin 3\theta + \dots \infty$$

$$(522) \sin \frac{\theta}{2} - \frac{1}{3^2} \sin \frac{3}{2} \theta + \frac{1}{5^2} \sin \frac{5}{2} \theta + \dots \infty$$

$$(523) \sin^2 \theta - \frac{1}{2} \sin^2 2\theta + \frac{1}{3} \sin^2 3\theta - \dots \infty$$

$$(524) \sin 2\theta + \frac{1}{2} \sin 3\theta + \frac{1}{3} \sin 4\theta + \dots \infty$$

Reference

$$= \frac{1}{8} \{2\sqrt{3} \log (2 + \sqrt{3}) - \pi\} \quad \text{E. 123}$$

$$= \theta \quad \text{where } \frac{\pi}{4} > \theta > -\frac{\pi}{4} \quad \text{E. 107}$$

$$= \frac{\pi \sinh \theta}{2 \sinh \pi} \quad \text{U. 42}$$

$$= \frac{1}{4} (\theta - \pi)^2 - \frac{1}{12} \pi^2 \quad \text{where } 0 \leq \theta \leq 2\pi \quad \text{A. 360}$$

$$= \frac{\pi^2}{12} - \frac{\theta^2}{4} \quad \text{where } -\pi \leq \theta \leq \pi \quad \text{A. 360}$$

$$= \frac{\pi}{8} (\pi - 2\theta) \quad \text{where } 0 \leq \theta \leq \pi \quad \text{A. 360}$$

$$= \frac{\pi}{8} (\pi + 2\theta) \quad \text{where } -\pi \leq \theta \leq 0 \quad \text{A. 360}$$

$$= \frac{1}{4} \pi \theta \quad \text{where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad \text{A. 360}$$

$$= \frac{\pi}{4} (\pi - \theta) \quad \text{where } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \quad \text{A. 360}$$

$$= \frac{1}{2} \theta (\pi - \theta) \quad \text{where } 0 \leq \theta \leq \pi \quad \text{Q. 163}$$

$$= \frac{1}{2} \{\pi |\theta| - \theta^2\} \quad \text{where } -\pi \leq \theta \leq \pi \quad \text{Q. 163}$$

$$= \frac{4 \sin \theta}{5 - 4 \cos \theta} \quad \text{E. 117}$$

$$= \frac{\pi \theta}{8} \quad \text{J. 311}$$

$$= \frac{1}{2} \log \sec \theta \quad \text{C. 334}$$

$$= -\sin \theta \log \left(4 \sin^2 \frac{\theta}{2} \right) + \frac{1}{2} (\pi - \theta) \cos \theta$$

Series No.

$$(525) \sin \theta + \frac{1}{2^3} \sin 2\theta + \frac{1}{3^3} \sin 3\theta + \dots \infty$$

$$(526) \sin \theta - \frac{1}{2^3} \sin 2\theta + \frac{1}{3^3} \sin 3\theta - \dots \infty$$

$$(527) \sin \theta + \frac{1}{3^3} \sin 3\theta + \frac{1}{5^3} \sin 5\theta + \dots \infty$$

$$(528) \cos \theta - \frac{1}{3^3} \cos 3\theta + \frac{1}{5^3} \cos 5\theta - \dots \infty$$

$$(529) \cos \theta + \frac{1}{2^4} \cos 2\theta + \frac{1}{3^4} \cos 3\theta + \dots \infty$$

$$(530) \cos \theta - \frac{1}{2^4} \cos 2\theta + \frac{1}{3^4} \cos 3\theta - \dots \infty$$

$$(531) \cos \theta + \frac{1}{3^4} \cos 3\theta + \frac{1}{5^4} \cos 5\theta + \dots \infty$$

$$(532) \sin \theta - \frac{1}{3^4} \sin 3\theta + \frac{1}{5^4} \sin 5\theta - \dots \infty$$

$$(533) \sin \theta + \frac{1}{2^5} \sin 2\theta + \frac{1}{3^5} \sin 3\theta + \dots \infty$$

$$(534) \sum_{-\infty}^{+\infty} \frac{\cos n\theta}{n-a}$$

$$(535) \sum_{-\infty}^{+\infty} \frac{\sin n\theta}{n-a}$$

$$(536) a \cos \theta + \frac{a^2}{2} \cos 2\theta + \frac{a^3}{3} \cos 3\theta + \dots \infty$$

$$(537) a \cos \theta + \frac{a^3}{3} \cos 3\theta + \frac{a^5}{5} \cos 5\theta + \dots \infty$$

Reference

$$= \frac{1}{12} \{(\theta - \pi)^3 - \pi^2\theta + \pi^3\} \quad \text{where } 0 \leq \theta \leq 2\pi \quad \text{A. 362}$$

$$= \frac{1}{12} (\pi^2\theta - \theta^3) \quad \text{where } -\pi \leq \theta \leq \pi \quad \text{A. 362}$$

$$= \frac{1}{8} (\pi^2\theta - \pi\theta^2) \quad \text{where } 0 \leq \theta \leq \pi$$

$$= \frac{1}{8} (\pi^2\theta + \pi\theta^2) \quad \text{where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad \text{A. 362}$$

$$= \frac{1}{8} \pi \left(\frac{1}{4} \pi^2 - \theta^2 \right) \quad \text{where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad \text{A. 362}$$

$$= \frac{1}{48} \{2\pi^2(\theta - \pi)^2 - (\theta - \pi)^4\} - \frac{7\pi^4}{720} \quad \text{where } 0 \leq \theta \leq 2\pi \quad \text{A. 363}$$

$$= \frac{1}{48} \left\{ \theta^4 - 2\pi^2\theta^2 + \frac{7}{15} \pi^4 \right\} \quad \text{where } -\pi \leq \theta \leq \pi \quad \text{A. 363}$$

$$= \frac{\pi}{96} \{4\theta^3 - 6\pi\theta^2 + \pi^3\} \quad \text{where } 0 \leq \theta \leq \pi \quad \text{A. 363}$$

$$= \frac{\pi\theta}{96} \{3\pi^2 - 4\theta^2\} \quad \text{where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad \text{A. 363}$$

$$= \frac{\pi^4\theta}{90} - \frac{\pi^2\theta^3}{36} + \frac{\pi\theta^4}{48} - \frac{\theta^5}{240} \quad \text{where } 0 < \theta < 2\pi \quad \text{T. 138}$$

$$= -\frac{\pi \cos a(\pi - \theta)}{\sin a\pi} \quad \text{where } 0 < \theta < 2\pi \quad \text{A. 370}$$

$$= \frac{\pi \sin a(\pi - \theta)}{\sin a\pi} \quad \text{where } 0 < \theta < 2\pi \quad \text{A. 370}$$

$$= -\frac{1}{2} \log (1 - 2a \cos \theta + a^2) \quad \text{where } a^2 < 1 \text{ and } \theta \neq 2n\pi \quad \text{2Z. 302}$$

$$= \frac{1}{4} \log \frac{1 + 2a \cos \theta + a^2}{1 - 2a \cos \theta + a^2} \quad \text{where } |a| < 1 \quad \text{E. 122}$$

Series No.

$$(538) \log h a^2 - 2 \left(\frac{1}{a} \cos \theta + \frac{1}{2a^2} \cos 2\theta + \frac{1}{3a^3} \cos 3\theta + \dots \infty \right)$$

$$(539) a \cos \theta - \frac{a^3}{3} \cos 3\theta + \frac{a^5}{5} \cos 5\theta - \dots \infty$$

$$(540) a \sin \theta + \frac{a^2}{2} \sin 2\theta + \frac{a^3}{3} \sin 3\theta + \dots \infty$$

$$(541) (\pi - \theta) - \left(\frac{1}{a} \sin \theta + \frac{1}{2a^2} \sin 2\theta + \frac{1}{3a^3} \sin 3\theta + \dots \infty \right)$$

$$(542) a \sin \theta - \frac{a^2}{2} \sin 2\theta + \frac{a^3}{3} \sin 3\theta - \dots \infty$$

$$(543) a \sin \theta + \frac{a^3}{3} \sin 3\theta + \frac{a^5}{5} \sin 5\theta + \dots \infty$$

$$(544) a \sin \theta - \frac{a^3}{3} \sin 3\theta + \frac{a^5}{5} \sin 5\theta - \dots \infty$$

$$(545) \frac{1}{a} \cos \theta + \frac{1}{a^2} \cos 2\theta + \frac{1}{a^3} \cos 3\theta + \dots \infty$$

$$(546) \sum_1^{\infty} \frac{a^{2n-1}}{2n-1} \sin (2n-1)\theta$$

$$(547) \sum_1^{\infty} (-1)^{n-1} \frac{a^{2n-1}}{2n-1} \cos (2n-1)\theta$$

$$(548) 2a \sin \theta + 4a^2 \sin 2\theta + 6a^3 \sin 3\theta + \dots \infty$$

$$(549) \sum_0^{\infty} \frac{\cos (n + \frac{1}{2})\theta}{n + \frac{1}{2}}$$

$$(550) \sum_0^{\infty} \frac{\sin (n + \frac{1}{2})\theta}{n + \frac{1}{2}}$$

$$(551) \sum_{-\infty}^{+\infty} \frac{\sin (n-a)\theta}{n-a}$$

Reference

$$= \log h (1 - 2a \cos \theta + a^2) \quad \text{where } a^2 > 1 \quad 2Z. 302$$

$$= \frac{1}{2} \tan^{-1} \frac{2a \cos \theta}{1 - a^2} \quad \text{where } |a| < 1 \quad E. 122$$

$$= \tan^{-1} \frac{a \sin \theta}{1 - a \cos \theta} \quad \text{where } a^2 < 1 \quad 2Z. 302$$

$$= \tan^{-1} \frac{a \sin \theta}{1 - a \cos \theta} \quad \text{where } a^2 > 1 \quad 2Z. 302$$

$$= \tan^{-1} \frac{a \sin \theta}{1 + a \cos \theta} \quad \text{where } |a| < 1, \quad \theta \neq (2n + 1)\pi \quad E. 122$$

$$= \frac{1}{2} \tan^{-1} \frac{2a \sin \theta}{1 - a^2} \quad \text{where } |a| < 1, \quad \theta \neq n\pi \quad E. 122$$

$$= \frac{1}{4} \log h \frac{1 + 2a \sin \theta + a^2}{1 - 2a \sin \theta + a^2} \quad \text{where } |a| < 1 \quad E. 122$$

$$= \frac{a \cos \theta - 1}{1 - 2a \cos \theta + a^2} \quad \text{where } a^2 > 1 \quad 2Z. 302$$

$$= \frac{1}{2} \tan^{-1} \frac{2a \sin \theta}{1 - a^2} \quad \text{where } a^2 < 1 \quad T. 139$$

$$= \frac{1}{2} \tan^{-1} \frac{2a \cos \theta}{1 - a^2} \quad \text{where } a^2 < 1 \quad T. 140$$

$$= \frac{2a(1 - a^2) \sin \theta}{(1 - 2a \cos \theta + a^2)^2} \quad \text{where } a^2 < 1 \quad 2Z. 303$$

$$= \log h \cot \frac{\theta}{4} \quad \text{where } 0 < \theta < \pi \quad A. 392$$

$$= \frac{\pi}{2} \quad \text{where } 0 < \theta < \pi \quad A. 392$$

$$= \pi \quad \text{where } 0 < \theta < 2\pi \quad A. 371$$

Series No.

$$(552) \sum_{-\infty}^{+\infty} \frac{\cos (n-a) \theta}{n-a}$$

$$(553) 2m \sum_1^{\infty} \frac{\cos n \theta}{n^2 - m^2}$$

$$(554) 2 \sum_1^{\infty} \frac{n \sin n \theta}{n^2 - m^2}$$

$$(555) \sum_1^{\infty} \frac{\cos n \theta}{n^2 - a^2}$$

$$(556) \sum_1^{\infty} \frac{n \sin n \theta}{n^2 - a^2}$$

$$(557) \sum_1^{\infty} (-1)^{n-1} \frac{n \sin n \theta}{n^2 - a^2}$$

$$(558) a \sum_1^{\infty} (-1)^{n-1} \frac{\cos n \theta}{n^2 - a^2}$$

$$(559) 2a \sum_1^{\infty} \frac{\cos n \theta}{n^2 + a^2}$$

$$(560) 2 \sum_1^{\infty} \frac{n \sin n \theta}{n^2 + a^2}$$

$$(561) \sum_1^{\infty} (-1)^{n-1} \frac{n \sin n \theta}{n^2 + a^2}$$

$$(562) a \sum_1^{\infty} (-1)^n \frac{\cos n \theta}{n^2 + a^2}$$

$$(563) \sum_2^{\infty} \frac{(-1)^n}{n} \left[\frac{\cos 2(n-1)\theta}{(n-1)^2} - \frac{\cos 2(n+1)\theta}{(n+1)^2} \right]$$

Reference

$$= -\pi \cot \pi a \quad \text{where } 0 < \theta < 2\pi \quad \text{A. 371}$$

$$= \frac{1}{m} + \frac{\cos m\theta}{2m} - (\pi - \theta) \sin m\theta \quad \text{where } 0 < \theta < 2\pi \text{ and } m \text{ is a positive integer (omit } m = n) \quad \text{A. 371}$$

$$= (\pi - \theta) \cos m\theta - \frac{\sin m\theta}{2m} \quad \text{where } 0 < \theta < 2\pi \text{ and } m \text{ is a positive integer (omit } m = n) \quad \text{A. 371}$$

$$= \frac{\pi \sin a(\frac{1}{2}\pi - \theta)}{4a \cos \frac{1}{2}\pi a} \quad \text{where } 0 < \theta < \pi \text{ and } n \text{ is odd (} a \text{ unrestricted)} \quad \text{A. 371}$$

$$= \frac{\pi \cos a(\frac{1}{2}\pi - \theta)}{4 \cos \frac{1}{2}\pi a} \quad \text{where } 0 < \theta < \pi \text{ and } n \text{ is odd (} a \text{ unrestricted)} \quad \text{A. 371}$$

$$= \frac{\pi \sin a\theta}{2 \sin \pi a} \quad \text{where } -\pi < \theta < \pi \quad \text{Q. 191}$$

$$= \frac{\pi \cos a\theta}{2 \sin \pi a} - \frac{1}{2a} \quad \text{where } -\pi < \theta < \pi \quad \text{Q. 191}$$

$$= \frac{\pi \cosh a(\pi - \theta)}{\sinh \pi a} - \frac{1}{a} \quad \text{where } 0 < \theta < 2\pi \quad \begin{cases} \text{A. 393} \\ \text{AE. 5} \end{cases}$$

$$= \frac{\pi \sinh a(\pi - \theta)}{\sinh \pi a} \quad \text{where } 0 < \theta < 2\pi \quad \begin{cases} \text{A. 393} \\ \text{AE. 4} \end{cases}$$

$$= \frac{\pi \sinh a\theta}{2 \sinh \pi a} \quad \text{where } -\pi < \theta < \pi \quad \text{2Z. 717}$$

$$= + \frac{\pi \cosh a\theta}{2 \sinh \pi a} - \frac{1}{2a} \quad \text{where } -\pi < \theta < \pi \quad \text{2Z. 717}$$

$$= 1 + \frac{\pi^2}{6} - 2\theta^2 - 2 \cos 2\theta - 2\theta \sin 2\theta - \frac{1}{4} \cos 4\theta$$

Series No.

$$(564) \frac{a}{\pi} \sum_1^{\infty} \frac{\sin \frac{2n\pi\theta}{a}}{n}$$

See No. (572)

$$(565) \frac{a^2}{\pi^2} \sum_1^{\infty} \frac{\cos \frac{2n\pi\theta}{a}}{n^2}$$

See No. (573)

$$(566) \frac{3}{4} \sin \theta + 3 \sin^3 \frac{\theta}{3} + 3^2 \sin^3 \frac{\theta}{3^2} + \dots \infty$$

$$(567) \sin^2 \theta + 2^2 \sin^4 \frac{\theta}{2} + 2^4 \sin^4 \frac{\theta}{2^2} + 2^6 \sin^4 \frac{\theta}{2^3} + \dots \infty$$

$$(568) \cos 2\theta - \frac{1}{2} \cos 4\theta + \frac{1}{3} \cos 6\theta - \frac{1}{4} \cos 8\theta \\ + \frac{1}{5} \cos 10\theta + \dots \infty$$

$$(569) 1 - \frac{1}{4} \tan^2 2\theta + \frac{1}{8} \tan^4 2\theta - \frac{5}{64} \tan^6 2\theta + \dots \infty$$

$$(570) 1 - \frac{3}{8} \tan^2 2\theta + \frac{31}{128} \tan^4 2\theta - \frac{187}{1024} \tan^6 2\theta + \dots \infty$$

$$(571) 1 - \frac{11}{32} \tan^2 2\theta + \frac{431}{2048} \tan^4 2\theta - \dots \infty$$

$$(572) \dagger 2 \sum_1^{\infty} \frac{\sin 2n\pi\theta}{2n\pi}$$

$$(573) \dagger 2 \sum_1^{\infty} \frac{\cos 2n\pi\theta}{(2n\pi)^2}$$

$$(574) \dagger 2 \sum_1^{\infty} \frac{\sin 2n\pi\theta}{(2n\pi)^{2k-1}}$$

$$(575) \dagger 2 \sum_1^{\infty} \frac{\cos 2n\pi\theta}{(2n\pi)^{2k}}$$

† For $\phi_k(\theta)$ see No. (1128). B_k are Bernoulli numbers.

Reference

$$= \frac{1}{2} a - \theta \quad \text{where } 0 < \theta < a$$

$$= \left(\frac{1}{2} a - \theta \right)^2 - \frac{a^2}{12} \quad \text{where } 0 \leq \theta \leq a$$

$$= \frac{3}{4} \theta \quad \text{C. 336}$$

$$= \theta^2 \quad \text{C. 336}$$

$$= \log h 2 + \log h \cos \theta \quad \text{where } -\pi < \theta < \pi \quad \text{2Z. 302}$$

$$= \frac{2 \tan \theta}{\tan 2\theta} \quad \text{O. 383}$$

$$= \frac{2 \sin \theta}{\tan 2\theta} \quad \text{O. 383}$$

$$= \frac{4 \sin \frac{\theta}{2}}{\tan 2\theta} \quad \text{O. 383}$$

$$= \frac{1}{2} - \phi_1(\theta) = \frac{1}{2} - \theta \quad \text{where } 0 < \theta < 1 \quad \begin{cases} \text{A. 370} \\ \text{AE. 2} \end{cases}$$

$$= \frac{1}{2!} \{ \phi_2(\theta) + B_1 \} = \frac{1}{2} \left(\theta^2 - \theta + \frac{1}{6} \right) \quad \text{where } 0 < \theta < 1 \quad \begin{cases} \text{A. 370} \\ \text{AE. 2} \end{cases}$$

$$= \frac{(-1)^k}{(2k-1)!} \phi_{2k-1}(\theta) \quad \text{where } k > 1 \text{ and } 0 < \theta < 1 \quad \text{A. 370}$$

$$= \frac{(-1)^{k-1}}{(2k)!} \{ \phi_{2k}(\theta) + (-1)^{k-1} B_k \} \quad \text{where } 0 < \theta < 1 \quad \text{A. 370}$$

Series No.

$$(576)^\dagger \sin 2\pi\theta + \frac{\sin 4\pi\theta}{2^{2n+1}} + \frac{\sin 6\pi\theta}{3^{2n+1}} + \dots \infty$$

$$(577)^\dagger \cos 2\pi\theta + \frac{\cos 4\pi\theta}{2^{2n}} + \frac{\cos 6\pi\theta}{3^{2n}} + \dots \infty$$

$$(578) \sum_1^\infty \tan^{-1} \frac{2\theta^2}{n^2}$$

$$(579) \sum_1^\infty (-1)^{n-1} \tan^{-1} \frac{2\theta^2}{n^2}$$

$$(580) \tan^{-1} \frac{\theta}{1-\theta^2} - \tan^{-1} \frac{3\theta}{3^2-\theta^2} + \tan^{-1} \frac{5\theta}{5^2-\theta^2} - \dots \infty$$

$$(581) \frac{1}{2} \tan \frac{\theta}{2} + \frac{1}{2^2} \tan \frac{\theta}{2^2} + \frac{1}{2^3} \tan \frac{\theta}{2^3} + \dots \infty$$

$$(582) \frac{1}{2^2} \sec^2 \frac{\theta}{2} + \frac{1}{2^4} \sec^2 \frac{\theta}{2^2} + \frac{1}{2^6} \sec^2 \frac{\theta}{2^3} + \dots \infty$$

$$(583) \left\{ \frac{\pi^2}{1} - \frac{4}{1^3} \right\} \sin \theta - \frac{\pi^2}{2} \sin 2\theta + \left\{ \frac{\pi^2}{3} - \frac{4}{3^3} \right\} \sin 2\theta \\ - \frac{\pi^2}{4} \sin 4\theta + \dots \infty$$

$$(584) \left\{ \frac{\pi^3}{1} - \frac{6\pi}{1^3} \right\} \sin \theta - \left\{ \frac{\pi^3}{2} - \frac{6\pi}{2^3} \right\} \sin 2\theta \\ + \left\{ \frac{\pi^3}{3} - \frac{6\pi}{3^3} \right\} \sin 3\theta - \dots \infty$$

$$(585) \frac{1}{2} (1 + \epsilon^n) \sin \theta + \frac{2}{5} (1 - \epsilon^n) \sin 2\theta + \frac{3}{10} (1 + \epsilon^n) \sin 3\theta \\ + \frac{4}{17} (1 - \epsilon^n) \sin 4\theta - \dots \infty$$

$$(586) \left\{ \frac{\pi^2}{1^2} - \frac{4}{1^4} \right\} \cos \theta - \frac{\pi^2}{2^2} \cos 2\theta + \left\{ \frac{\pi^2}{3^2} - \frac{4}{3^4} \right\} \cos 3\theta \\ - \frac{\pi^2}{4^2} \cos 4\theta + \left\{ \frac{\pi^2}{5^2} - \frac{4}{5^4} \right\} \cos 5\theta - \dots \infty$$

† For values of $B_n(\theta)$, see No. (1142).

Reference

$$= (-1)^{n+1} \frac{2^{2n} \pi^{2n+1}}{(2n)!} B_{2n+1}(\theta) \quad \text{where } 0 < \theta < 1 \quad \text{AE. 3}$$

$$= (-1)^{n-1} \frac{2^{2n-1} \pi^{2n}}{(2n-1)!} \left\{ B_{2n}(\theta) + (-1)^{n-1} \frac{B_n}{2n} \right\} \quad \text{where } 0 < \theta < 1 \quad \text{AE. 3}$$

$$= \frac{\pi}{4} - \tan^{-1} \frac{\tanh \pi \theta}{\tan \pi \theta} \quad \text{A. 314}$$

$$= -\frac{\pi}{4} + \tan^{-1} \frac{\sinh \pi \theta}{\sin \pi \theta} \quad \text{A. 314}$$

$$= \tan^{-1} \frac{\sinh \frac{\pi \theta}{4}}{\cos \frac{\sqrt{3} \pi \theta}{4}} \quad \text{A. 314}$$

$$= \frac{1}{\theta} - \cot \theta \quad \text{Y. 54}$$

$$= \operatorname{cosec}^2 \theta - \frac{1}{\theta^2} \quad \text{Y. 54}$$

$$= \frac{\pi \theta^2}{2} \quad \text{U. 41}$$

$$= \frac{\pi \theta^3}{2} \quad \text{U. 41}$$

$$= \frac{\pi \epsilon^\theta}{2} \quad \text{U. 42}$$

$$= \left\{ \frac{\pi^3}{4} - \theta^3 \right\} \frac{\pi}{6} \quad \text{U. 45}$$

Series No.

$$(587) \quad \epsilon^a \cos \theta - \frac{1}{3} \epsilon^{3a} \cos 3\theta + \frac{1}{5} \epsilon^{5a} \cos 5\theta - \dots \infty$$

$$(588) \quad \frac{8a}{\pi^2} \sum_0^\infty \left\{ \frac{\cos(2n+1) \frac{\pi\theta}{2a}}{2n+1} \right\}^2$$

$$(589) \quad \sum_1^\infty \frac{n}{n^2 + a^2} \{1 + (-1)^{n-1} \epsilon^{an}\} \sin n\theta$$

$$(590) \quad 1 + a^2 + 4a \cos \theta + 2a^2(3 - a^2) \cos 2\theta \\ + 2a^3(4 - 2a^2) \cos 3\theta + \dots \\ + 2a^n\{n(1 - a^2) + (1 + a^2)\} \cos n\theta + \dots \infty$$

$$(591) \quad a^2 + 1 + 4a \cos \theta + \frac{2(3a^2 - 1)}{a^2} \cos 2\theta + \dots \\ + \frac{2\{n(a^2 - 1) + (a^2 + 1)\}}{a^n} \cos n\theta + \dots \infty$$

$$(592) \quad \sum_1^\infty \frac{na^{n-1}}{2(1 - a^2)^3} [n(1 - a^2) + (1 + a^2)] \sin \theta$$

$$(593) \quad \sum_1^\infty \frac{n}{2(a^2 - 1)^3} \frac{1}{a^{n+1}} [n(a^2 - 1) + (a^2 + 1)] \sin n\theta$$

Memorandum: Consult Edwards *Integral Calculus*, Vol. II, for extension to these series.

$$(594) \quad \frac{\epsilon^{-q\theta} \sin \theta}{1} + \frac{\epsilon^{-2q\theta} \sin 2\theta}{2} + \frac{\epsilon^{-3q\theta} \sin 3\theta}{3} + \dots \infty$$

$$(595) \quad \frac{\epsilon^{-q\theta} \cos \theta}{1} + \frac{\epsilon^{-2q\theta} \cos 2\theta}{2} + \frac{\epsilon^{-3q\theta} \cos 3\theta}{3} + \dots \infty$$

$$(596) \quad \frac{4}{3} \sin 2\theta + \frac{8}{15} \sin 4\theta + \dots + \frac{1 + (-1)^n}{n^2 - 1} n \sin n\theta + \dots \infty$$

Reference

$$= -\frac{1}{2} \tan^{-1} (\cos \theta \operatorname{cosech} a)$$

E. 123

$$= a - \theta \quad \text{where } \theta \text{ is positive, } |\theta| < a$$

J. 309

$$= a + \theta \quad \text{where } \theta \text{ is negative, } |\theta| < a$$

$$= \frac{\pi}{2} \epsilon^{a\theta} \quad \text{where } 0 < \theta < \pi$$

U. 42

$$= \frac{(1 - a^2)^3}{(1 - 2a \cos \theta + a^2)^2} \quad \text{where } a^2 < 1$$

2Z. 303

$$= \frac{(a^2 - 1)^3}{(1 - 2a \cos \theta + a^2)^2} \quad \text{where } a^2 > 1$$

2Z. 304

$$= \frac{\sin \theta}{(1 - 2a \cos \theta + a^2)^3} \quad \text{where } a^2 < 1$$

2Z. 305

$$= \frac{\sin \theta}{(1 - 2a \cos \theta + a^2)^3} \quad \text{where } a^2 > 1$$

2Z. 305

$$= \tan^{-1} \frac{\sin \theta}{\epsilon^{q\theta} - \cos \theta} \quad \text{where } q \text{ is positive}$$

2Z. 194

$$= -\log \sqrt{1 - 2\epsilon^{-q\theta} \cos \theta + \epsilon^{-2q\theta}} \quad \text{where } q \text{ is positive}$$

2Z. 194

$$= \frac{\pi \cos \theta}{2} \quad \text{where } 0 < \theta < \pi$$

Series No.

$$(597) \frac{1}{2}(1 + \cosh \pi) \sin \theta + \frac{2}{5}(1 - \cosh \pi) \sin 2\theta \\ + \frac{3}{10}(1 + \cosh \pi) \sin 3\theta + \dots \infty$$

$$(598) \frac{1 + e^a}{a^2 + \pi^2} \sin \frac{\pi\theta}{a} + \frac{2(1 - e^a)}{a^2 + 4\pi^2} \sin \frac{2\pi a}{a} \\ + \frac{3(1 + e^a)}{a^2 + 9\pi^2} \sin \frac{3\pi\theta}{a} + \dots \infty$$

$$(599) \sum_1^{\infty} \frac{(-1)^{n-1}}{n^2} \cos \frac{n\pi\theta}{a}$$

$$(600) \sum_1^{\infty} \frac{\cos 2n\theta}{(2n-1)(2n+1)}$$

$$(601) \frac{6}{\pi} \sum_1^{\infty} \frac{\sin \frac{1}{3}(2n-1)\pi \cdot \sin (2n-1)\theta}{(2n-1)^2}$$

$$(602) \sum_1^{\infty} \frac{\cos (2n+2)\theta}{n(n+1)}$$

$$(603) \sum_1^{\infty} \frac{\sin (2n+2)\theta}{n(n+1)}$$

$$(604) \frac{\sin 2\theta}{1 \cdot 3} + \frac{\sin 3\theta}{2 \cdot 4} + \dots \infty = \sum_2^{\infty} \frac{\sin n\theta}{n^2 - 1}$$

$$(605) \frac{\cos 2\theta}{1 \cdot 3} + \frac{\cos 3\theta}{2 \cdot 4} + \dots \infty = \sum_2^{\infty} \frac{\cos n\theta}{n^2 - 1}$$

$$(606) \frac{\cos 2\theta}{1 \cdot 3} - \frac{\cos 4\theta}{3 \cdot 5} + \dots \infty = - \sum_2^{\infty} \frac{\cos \frac{n\pi}{2} \cos n\theta}{n^2 - 1}$$

$$= \frac{\pi \cosh \theta}{2} \quad \text{U. 42}$$

$$= \frac{\epsilon^\theta}{2\pi} \quad \text{where } 0 < \theta < a \quad \text{U. 51}$$

$$= \frac{\pi^2}{4a^2} \left(\frac{a^2}{3} - \theta^2 \right) \quad \text{where } -a \leq \theta \leq a \quad \text{T. 139}$$

$$= \frac{1}{2} - \frac{\pi}{4} \sin \theta \quad \text{where } 0 \leq \theta \leq \frac{\pi}{2} \quad \text{T. 139}$$

$$= \frac{3}{2} \theta \quad \text{where } 0 \leq \theta \leq \frac{1}{3} \pi$$

$$= \frac{\pi}{2} \quad \text{where } \frac{1}{3} \pi \leq \theta \leq \frac{2}{3} \pi \quad \text{AB. 243}$$

$$= \frac{3}{2} (\pi - \theta) \quad \text{where } \frac{2}{3} \pi \leq \theta \leq \pi$$

$$= \cos 2\theta - \left(\frac{\pi}{2} - \theta \right) \sin 2\theta + \sin^2 \theta \log h (4 \sin^2 \theta) \quad \text{where } 0 \leq \theta \leq \pi \quad \text{T. 139}$$

$$= \sin 2\theta - (\pi - 2\theta) \sin^2 \theta - \sin \theta \cos \theta \log h (4 \sin^2 \theta) \quad \text{where } 0 \leq \theta \leq \pi \quad \text{T. 139}$$

$$= \frac{1}{4} \sin \theta - \sin \theta \log h \left(2 \sin \frac{\theta}{2} \right) \quad \text{where } 0 < \theta < 2\pi \quad \text{A. 368}$$

$$= \frac{1}{2} + \frac{1}{4} \cos \theta - \frac{1}{2} (\pi - \theta) \sin \theta \quad \text{where } 0 < \theta < 2\pi \quad \text{A. 368}$$

$$= \frac{\pi}{4} \cos \theta - \frac{1}{2}$$

Series No.

$$(607) \quad \frac{2 \sin 2\theta}{1 \cdot 3} - \frac{4 \sin 4\theta}{3 \cdot 5} + \dots \infty = - \sum_2^{\infty} \frac{\cos \frac{n\pi}{2} n \sin n\theta}{n^2 - 1}$$

$$(608) \quad \frac{\cos 2\theta}{1 \cdot 3} - \frac{\cos 3\theta}{2 \cdot 4} + \dots \infty = + \sum_2^{\infty} \frac{(-1)^n \cos n\theta}{n^2 - 1}$$

$$(609) \quad \frac{1}{2} - \frac{\cos 2\theta}{1 \cdot 3} - \frac{\cos 4\theta}{3 \cdot 5} - \frac{\cos 6\theta}{5 \cdot 7} - \dots \infty$$

$$(610) \quad \frac{1}{2} + \frac{1}{4} \cos \theta - \frac{1}{1 \cdot 3} \cos 2\theta - \frac{1}{2 \cdot 4} \cos 3\theta - \frac{1}{3 \cdot 5} \cos 4\theta - \dots \infty$$

$$(611) \quad \frac{\cos 3\theta}{2 \cdot 4} + \frac{\cos 5\theta}{4 \cdot 6} + \dots \infty = \sum_2^{\infty} \frac{\sin^2 \frac{n\pi}{2} \cos n\theta}{n^2 - 1}$$

$$(612) \quad \frac{\sin 3\theta}{2 \cdot 4} - \frac{\sin 5\theta}{4 \cdot 6} + \dots \infty = - \sum_2^{\infty} \frac{\sin \frac{n\pi}{2} \sin n\theta}{n^2 - 1}$$

$$(613) \quad \frac{\sin 3\theta}{2 \cdot 4} + \frac{\sin 5\theta}{4 \cdot 6} + \dots \infty = \sum_2^{\infty} \frac{\sin^2 \frac{n\pi}{2} \sin n\theta}{n^2 - 1}$$

$$(614) \quad \frac{3 \cos 3\theta}{2 \cdot 4} - \frac{5 \cos 5\theta}{4 \cdot 6} + \dots \infty = - \sum_2^{\infty} \frac{\sin \frac{n\pi}{2} n \cos n\theta}{n^2 - 1}$$

$$(615) \quad \frac{2 \cos 2\theta}{1 \cdot 3} + \frac{3 \cos 3\theta}{2 \cdot 4} + \dots \infty = \sum_2^{\infty} \frac{n \cos n\theta}{n^2 - 1}$$

$$(616) \quad \sum_0^{\infty} \frac{(-1)^n \cos n\theta}{(n+1)(n+2)}$$

$$(617) \quad \frac{1}{2} \sin \theta + \frac{1 \cdot 3}{2 \cdot 4} \sin 2\theta + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \sin 3\theta + \dots \infty$$

$$= \frac{\pi}{4} \sin \theta$$

$$= \frac{1}{2} - \frac{1}{4} \cos \theta - \frac{1}{2} \theta \sin \theta$$

$$= \frac{\pi}{4} \sin \theta \quad \text{T. 139}$$

$$= \frac{1}{2} (\pi - \theta) \sin \theta \quad \text{where } 0 \leq \theta \leq \pi \quad \text{Q. 166}$$

$$= \frac{1}{2} (\pi - \theta) \sin \theta \quad \text{where } \pi \leq \theta \leq 2\pi$$

$$= -\frac{1}{2} (\pi + \theta) \sin \theta \quad \text{where } -\pi \leq \theta \leq 0$$

$$= \frac{1}{4} \cos \theta - \frac{\pi}{4} \sin \theta + \frac{1}{2} \theta \sin \theta$$

$$= \frac{1}{2} \theta \cos \theta - \frac{1}{4} \sin \theta$$

$$= \frac{1}{4} \sin \theta - \frac{1}{2} \sin \theta \log (2 \sin \theta)$$

$$= -\frac{1}{2} \theta \sin \theta + \frac{1}{4} \cos \theta$$

$$= -\frac{1}{2} - \frac{1}{4} \cos \theta - \cos \theta \log \left(2 \sin \frac{\theta}{2} \right)$$

$$= (\cos \theta + \cos 2\theta) \log \left(2 \cos \frac{\theta}{2} \right) + \frac{1}{2} \theta (\sin 2\theta + \sin \theta) - \cos \theta \quad \text{Q. 190}$$

$$= \left(2 \sin \frac{\theta}{2} \right)^{-1/2} \sin \frac{\pi - \theta}{4} \quad \text{E. 116}$$

Series No.

$$(618) \quad 1 + \frac{1}{2} \cos \theta + \frac{1 \cdot 3}{2 \cdot 4} \cos 2\theta + \dots \infty$$

$$(619) \quad \frac{\sin 2\theta}{1 \cdot 2 \cdot 3} + \frac{\sin 3\theta}{2 \cdot 3 \cdot 4} + \dots \infty = \sum_2^{\infty} \frac{\sin n\theta}{(n-1)n(n+1)}$$

$$(620) \quad \cos \theta + \frac{1}{2} \cdot \frac{\cos 3\theta}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\cos 5\theta}{5} + \dots \infty$$

$$(621) \quad \sin \theta + \frac{1}{2} \sin 3\theta + \frac{1 \cdot 3}{2 \cdot 4} \cdot \sin 5\theta + \dots \infty$$

$$(622) \quad 1 + \frac{1}{2} \cos 2\theta - \frac{1}{2 \cdot 4} \cos 4\theta + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \cos 6\theta - \dots \infty$$

$$(623) \quad \frac{\sin \theta}{3} - \frac{\sin 3\theta}{1 \cdot 3 \cdot 5} - \frac{\sin 5\theta}{3 \cdot 5 \cdot 7} - \dots \infty = - \sum_1^{\infty} \frac{\sin n\theta}{n(n^2 - 4)}$$

$$(624) \quad 1 - \frac{\sin^2 \theta}{3} - \frac{2}{3} \cdot \frac{\sin^4 \theta}{5} - \frac{2 \cdot 4}{3 \cdot 5} \frac{\sin^6 \theta}{7} - \dots \infty$$

$$(625) \quad \sin \theta + \frac{2}{3} \sin^3 \theta + \frac{2 \cdot 4}{3 \cdot 5} \sin^5 \theta + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \sin^7 \theta + \dots \infty$$

$$(626) \quad \frac{\cos \theta}{1 \cdot 2 \cdot 3} + \frac{\cos 2\theta}{2 \cdot 3 \cdot 4} + \frac{\cos 3\theta}{3 \cdot 4 \cdot 5} + \dots \infty$$

$$(627) \quad \frac{\cos 3\theta}{1 \cdot 3 \cdot 5} - \frac{\cos 5\theta}{3 \cdot 5 \cdot 7} + \frac{\cos 7\theta}{5 \cdot 7 \cdot 9} - \dots \infty$$

$$(628) \quad \frac{1}{2} \cdot \frac{\sin^2 \theta}{2} + \frac{1 \cdot 3}{2 \cdot 4} \frac{\sin^4 \theta}{4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{\sin^6 \theta}{6} + \dots \infty$$

$$(629) \quad 1 + \frac{2}{3} \sin^2 \theta + \frac{2 \cdot 4}{3 \cdot 5} \sin^4 \theta + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \sin^6 \theta + \dots \infty$$

$$= 1 + \frac{2^2}{3!} \sin^2 \theta + \frac{2^2 \cdot 4^2}{5!} \sin^4 \theta + \frac{2^2 \cdot 4^2 \cdot 6^2}{7!} \sin^6 \theta + \dots \infty$$

$$(630) \quad \sin \theta + a \cos \theta - \frac{a^2}{2!} \sin \theta - \frac{a^3}{3!} \cos \theta + \dots \infty$$

$$(631) \quad 1 + a \cos \theta + \frac{a^2}{2!} \cos 2\theta + \frac{a^3}{3!} \cos 3\theta + \dots \infty$$

Reference

$$= \left(2 \sin \frac{\theta}{2}\right)^{-1/2} \cos \frac{\pi - \theta}{4}$$

E. 116

$$= \frac{3}{4} \sin \theta - \frac{1}{2} (\pi - \theta) + \frac{1}{2} (\pi - \theta) \cos \theta$$

$$= \frac{1}{2} \cos^{-1} (1 - 2 \sin \theta)$$

C. 334

$$= \frac{1}{\sqrt{2 \sin \theta}} \sin \left(\frac{\pi}{4} + \frac{\theta}{2} \right)$$

E. 117

$$= \sqrt{\cos \theta (1 + \cos \theta)} \quad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

E. 118

$$= \frac{\pi}{8} \sin^2 \theta \quad \text{where } n \text{ is odd}$$

X. 54

$$= \theta \cot \theta$$

Y. 86

$$= \theta \sec \theta$$

Y. 505

$$= \frac{1}{2} \cos \theta - \frac{1}{4} \cos 2\theta$$

C. 421

$$= \frac{\pi}{8} \cos^2 \theta - \frac{1}{3} \cos \theta \quad \text{where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

A. 369

$$= \log h \sec^2 \frac{\theta}{2}$$

H. 498

$$= \frac{2\theta}{\sin 2\theta}$$

L. 78

$$= \sin (\theta + a)$$

Y. 84

$$= e^{a \cos \theta} \cos (a \sin \theta)$$

X. 72

Series No.

$$(632) \ a \sin \theta + \frac{a^2}{2!} \sin 2\theta + \frac{a^3}{3!} \sin 3\theta + \dots \infty$$

$$(633) \ 1 + \frac{a^2 \cos 2\theta}{2!} + \frac{a^4 \cos 4\theta}{4!} + \dots \infty$$

$$(634) \ \frac{a^2 \sin 2\theta}{2!} + \frac{a^4 \sin 4\theta}{4!} + \dots \infty$$

$$(635) \ \sum_0^{\infty} \frac{a^{2n} \cos 2n\theta}{(2n)!}$$

$$(636) \ \sum_0^{\infty} \frac{a^{2n+1} \cos (2n+1)\theta}{(2n+1)!}$$

$$(637) \ \sum_0^{\infty} \frac{a^{2n+1} \sin (2n+1)\theta}{(2n+1)!}$$

$$(638) \ \sum_1^{\infty} \frac{a^{2n} \sin 2n\theta}{(2n)!}$$

$$(639) \ 1 - \theta \sin \theta + \frac{\theta^2 \cos 2\theta}{2!} + \frac{2^2 \theta^3 \sin 3\theta}{3!} - \frac{3^3 \theta^4 \cos 4\theta}{4!} - \dots \infty$$

$$(640) \ \theta \cos \theta + \frac{\theta^2 \sin 2\theta}{2!} - \frac{2^2 \theta^3 \cos 3\theta}{3!} - \frac{3^3 \theta^4 \sin 4\theta}{4!} + \dots \infty$$

$$(641) \ r\theta \sin \phi + \frac{r^2 \theta^2}{2!} \sin 2\phi + \frac{r^3 \theta^3}{3!} \sin 3\phi + \dots \infty$$

$$\left(r = \sqrt{a^2 + b^2}, \right.$$

$$(642) \ 1 + \sin \theta + \frac{(1+1^2) \sin^2 \theta}{2!} + \frac{(1+2^2) \sin^3 \theta}{3!} + \dots \infty$$

$$(643) \ 1 + 2^{1/2} \cos \frac{\pi}{4} \theta + 2^{2/2} \cos \frac{2\pi}{4} \frac{\theta^2}{2!} + 2^{3/2} \cos \frac{3\pi}{4} \frac{\theta^3}{3!} + \dots \infty$$

$$= \sum_0^{\infty} 2^{n/2} \cos \frac{n\pi}{4} \cdot \frac{\theta^n}{n!}$$

$$(644) \ \cos \theta + n \cos 3\theta + \frac{n(n-1)}{2!} \cos 5\theta + \dots \infty$$

Reference

$$= \epsilon^{a \cos \theta} \sin (a \sin \theta) \quad \text{X. 72}$$

$$= \frac{1}{2} \cos (a \sin \theta) \{ \epsilon^{a \cos \theta} + \epsilon^{-a \cos \theta} \} \quad \text{E. 118}$$

$$= \frac{1}{2} \sin (a \sin \theta) \{ \epsilon^{a \cos \theta} - \epsilon^{-a \cos \theta} \} \quad \text{E. 118}$$

$$= \cosh (a \cos \theta) \cos (a \sin \theta) \quad \text{where } a^2 < 1 \quad \text{T. 127}$$

$$= \sinh (a \cos \theta) \cos (a \sin \theta) \quad \text{where } a^2 < 1 \quad \text{T. 127}$$

$$= \cosh (a \cos \theta) \sin (a \sin \theta) \quad \text{where } a^2 < 1 \quad \text{T. 127}$$

$$= \sinh (a \cos \theta) \sin (a \sin \theta) \quad \text{where } a^2 < 1 \quad \text{T. 127}$$

$$\left. \begin{aligned} &= \cos \theta \\ &= \sin \theta \end{aligned} \right\} \quad \text{where } |\theta| < \frac{1}{\epsilon} \quad \text{A. 312}$$

$$\phi = \tan^{-1} \frac{b}{a} = \epsilon^{a\theta} \sin b\theta \quad \text{E. 131}$$

$$= \frac{\epsilon^\theta}{\cos \theta} \quad \text{L. 79}$$

$$= \epsilon^\theta \cos \theta \quad \text{Y. 84}$$

$$= 2^n \cos^n \theta \cos (n+1)\theta \quad \text{X. 72}$$

Series No.

$$(645) \sin \theta + n \sin 3\theta + \frac{n(n-1)}{2!} \sin 5\theta + \dots \infty$$

$$(646) \frac{1}{2} + \frac{a}{a+2} \cos 2\theta + \frac{a(a-2)}{(a+2)(a+4)} \cos 4\theta + \dots \infty$$

$$(647) \cos^n \theta - \frac{n(n-1)}{2!} \cos^{n-2} \theta \sin^2 \theta \\ + \frac{n(n-1)(n-2)(n-3)}{4!} \cos^{n-4} \theta \sin^4 \theta - \dots \text{zero}$$

$$(648) n \cos^{n-1} \theta \sin \theta - \frac{n(n-1)(n-2)}{3!} \cos^{n-3} \theta \sin^3 \theta + \dots \text{zero}$$

$$(649) \cos n\theta + n \cos (n-2)\theta + \frac{n(n-1)}{2!} \cos (n-4)\theta + \dots$$

If n is odd there is an even number of terms so that the last term contains $\cos \theta$.

$$(650) \cos n\theta - n \cos (n-2)\theta + \frac{n(n-1)}{2!} \cos (n-4)\theta + \dots \\ + \frac{1}{2} (-1)^{n/2} \frac{n!}{\left\{ \left(\frac{n}{2} \right)! \right\}^2}$$

$$(651) \sin n\theta - n \sin (n-2)\theta + \frac{n(n-1)}{2!} \sin (n-4)\theta + \dots \\ + (-1)^{(n-1)/2} \frac{n!}{\left(\frac{n-1}{2} \right)! \left(\frac{n+1}{2} \right)!} \sin \theta$$

In the series Nos. (652) through (659) ${}_nC_r$ are the binomial coefficients.

$${}_nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

$$(652) \cos 2n\theta + {}_{2n}C_1 \cos (2n-2)\theta + {}_{2n}C_2 \cos (2n-4)\theta + \dots \\ + \frac{1}{2} {}_{2n}C_n$$

$$(653) \cos (2n+1)\theta + {}_{2n+1}C_1 \cos (2n-1)\theta + \dots + {}_{2n+1}C_n \cos \theta$$

Reference

$$= 2^n \cos^n \theta \sin (n+1)\theta \quad \text{X. 72}$$

$$= \frac{2^{a-1} \left\{ \Gamma\left(\frac{a}{2} + 1\right) \right\}^2}{\Gamma(a+1)} \cos^a \theta \quad \text{where } a \text{ is positive but not necessarily an integer and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad \text{Q. 263}$$

$$= \cos n\theta \quad \text{E. 33}$$

$$= \sin n\theta \quad \text{E. 33}$$

$$= 2^{n-1} \cos^n \theta \quad \text{E. 55}$$

$$= 2^{n-1} (-1)^{n/2} \sin^n \theta \quad \text{where } n \text{ is even} \quad \text{E. 57}$$

$$= 2^{n-1} (-1)^{(n-1)/2} \sin^n \theta \quad \text{where } n \text{ is odd} \quad \text{E. 58}$$

$$= 2^{2n-1} \cos^{2n} \theta \quad \text{where } n \text{ is even} \quad \text{C. 278}$$

$$= 2^{2n} \cos^{2n+1} \theta \quad \text{C. 278}$$

Series No.

$$(654) \cos 2n\theta - {}_{2n}C_1 \cos (2n-2)\theta + {}_{2n}C_2 \cos (2n-4)\theta + \dots \\ + (-1)^n \frac{1}{2} \cdot {}_{2n}C_n$$

$$(655) \sin (2n+1)\theta - {}_{2n+1}C_1 \sin (2n-1)\theta \\ + {}_{2n+1}C_2 \sin (2n-3)\theta + \dots + (-1)^n {}_{2n+1}C_n \sin \theta$$

$$(656) 1 + {}_nC_1 \cos 2\theta + {}_nC_2 \cos 4\theta + {}_nC_3 \cos 6\theta + \dots + \cos 2n\theta$$

$$(657) {}_nC_1 \sin 2\theta + {}_nC_2 \sin 4\theta + {}_nC_3 \sin 6\theta + \dots + \sin 2n\theta$$

$$(658) 1 - {}_nC_1 \cos 2\theta + {}_nC_2 \cos 4\theta - {}_nC_3 \cos 6\theta + \dots - \cos 2n\theta \\ {}_nC_1 \sin 2\theta - {}_nC_2 \sin 4\theta + {}_nC_3 \sin 6\theta - \dots - \sin 2n\theta$$

$$(659) {}_nC_1 \sin 2\theta - {}_nC_2 \sin 4\theta + {}_nC_3 \sin 6\theta - \dots + \sin 2n\theta \\ 1 - {}_nC_1 \cos 2\theta + {}_nC_2 \cos 4\theta - {}_nC_3 \cos 6\theta + \dots + \cos 2n\theta$$

$$(660) 1 - a\theta \sin b\theta - \frac{a(a-2b)}{2!} \theta^2 \cos 2b\theta \\ + \frac{a(a-3b)^2}{3!} \theta^3 \sin 3b\theta + \dots \infty$$

$$(661) \frac{1}{2} + \frac{2n-1}{2n+1} \cos 2\theta + \frac{(2n-1)(2n-3)}{(2n+1)(2n+3)} \cos 4\theta + \dots \infty$$

$$(662) \sin^2 \theta + \frac{2}{3} \cdot \frac{1}{2} \sin^4 \theta + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{1}{3} \sin^6 \theta + \dots \infty$$

$$(663) \cos \theta - \frac{1}{7} \cos 3\theta + \frac{1}{7} \cdot \frac{5}{11} \cos 5\theta - \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} \cos 7\theta + \dots \infty$$

$$\text{where } h = \frac{4}{\pi} \int_0^{\pi/2} \cos^{3/2} \theta \, d\theta$$

(664) General case of (663)

$$\cos \theta - \frac{1-n}{3+n} \cos 3\theta + \frac{1-n}{3+n} \cdot \frac{3-n}{5+n} \cos 5\theta \\ - \frac{1-n}{3+n} \cdot \frac{3-n}{5+n} \cdot \frac{5-n}{7+n} \cos 7\theta + \dots \infty$$

When n is odd, this series terminates.

Reference

$$= 2^{2n-1}(-1)^n \sin^{2n} \theta \quad \text{C. 278}$$

$$= 2^{2n}(-1)^n \sin^{2n+1} \theta \quad \text{C. 278}$$

$$= 2^n \cos n\theta \cos^n \theta \quad \text{where } n \text{ can be odd or even}$$

$$= 2^n \cos^n \theta \sin n\theta \quad \text{where } n \text{ can be odd or even}$$

$$= (-1)^{(n-1)/2} 2^n \sin^n \theta \sin n\theta \quad \text{where } n \text{ is odd}$$

$$= (-1)^{(n+2)/2} 2^n \sin^n \theta \sin n\theta \quad \text{where } n \text{ is even}$$

$$= (-1)^{(n-1)/2} 2^n \sin^n \theta \cos n\theta \quad \text{where } n \text{ is odd}$$

$$= (-1)^{n/2} 2^n \sin^n \theta \cos n\theta \quad \text{where } n \text{ is even}$$

$$= \cos a\theta \quad \text{Y. 511}$$

$$= \frac{\pi}{4} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n-2)} |\cos^{2n-1} \theta| \quad \text{where } n \text{ is an integer} \quad \text{Q. 191}$$

$$= \theta^2 \quad \text{Q. 130}$$

$$= \frac{1}{h} \cos^{1/2} \theta$$

$$= \frac{4}{\pi} \cdot \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} = 1.113 \quad \text{X. 56}$$

$$= \frac{1}{C} \cos^n \theta \quad \text{where } C = \frac{4}{\pi} \int_0^{\pi/2} \cos^{n+1} \theta \, d\theta = \frac{4}{\pi} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2} + \frac{3}{2}\right)}$$

where n is any value

X. 63

Series No.

$$(665) \sin^4 \theta + \frac{4}{3} \cdot \frac{2}{5} \left(1 + \frac{1}{2^2}\right) \sin^6 \theta + \dots$$

$$+ \frac{4 \cdot 6 \dots (2n-2)}{5 \cdot 7 \dots (2n-1)} \cdot \frac{2}{n} \left\{1 + \frac{1}{2^2} + \dots + \frac{1}{(n-1)^2}\right\} \sin^{2n} \theta + \dots \infty$$

$$(666) 1 + \frac{2^2}{3 \cdot 4} \sin^2 \theta + \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6} \sin^4 \theta$$

$$+ \frac{2^2 \cdot 4^2 \cdot 6^2}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \sin^6 \theta + \dots \infty$$

$$(667) \frac{1}{2} \cos^2 \theta - \frac{1}{3} \left(1 + \frac{1}{2}\right) \cos^3 \theta$$

$$+ \frac{1}{4} \left(1 + \frac{1}{2} + \frac{1}{3}\right) \cos^4 \theta - \dots \infty$$

$$(667a) \frac{1}{2} \cos 2\theta - \frac{1}{3} \left(1 + \frac{1}{2}\right) \cos 3\theta + \frac{1}{4} \left(1 + \frac{1}{2} + \frac{1}{3}\right) \cos 4\theta - \dots \infty$$

$$(668) \frac{\sin \theta}{2!} + 1^2 \cdot 3^2 \left(\frac{1}{1^2} + \frac{1}{3^2}\right) \frac{\sin^3 \theta}{4!}$$

$$+ 1^2 \cdot 3^2 \cdot 5^2 \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right) \frac{\sin^5 \theta}{6!} + \dots \infty$$

$$(669) \frac{2}{\pi} \sum_1^{\infty} \frac{n}{k^2 + n^2} (1 - \epsilon^{k\pi} \cos n\pi) \sin n\theta$$

$$(670) \frac{\epsilon^{k\pi} - 1}{k\pi} + \frac{2k}{\pi} \sum_1^{\infty} \frac{\epsilon^{k\pi} \cos n\pi - 1}{k^2 + n^2} \cos n\theta$$

$$(671) \sum_{r=0}^{r=n-1} \frac{x - a \cos \frac{2r\pi + \theta}{n}}{x^2 - 2ax \cos \frac{2r\pi + \theta}{n} + a^2}$$

$$(672) \tan^{-1} x + (h \sin \theta) \sin \theta - \frac{(h \sin \theta)^2}{2!} \sin 2\theta + \dots \infty$$

$$(673) 1 + \frac{\tan^2 \theta}{3} - \frac{2 \tan^4 \theta}{3 \cdot 5} + \frac{2 \cdot 4 \tan^6 \theta}{3 \cdot 5 \cdot 7} + \dots \infty$$

$$= \theta^4 \quad \text{C. 335}$$

$$= \left(\frac{\theta}{\sin \theta} \right)^2 \quad \text{L. 79}$$

$$= \frac{1}{2} \left[\log h 2 \cos^2 \frac{\theta}{2} \right]^2 \quad \text{where } -\pi < \theta < \pi \quad \text{C. 252}$$

$$= \frac{1}{8} \left[\log h 4 \cos^2 \frac{\theta}{2} \right]^2 - \frac{\theta^2}{8} \quad \text{where } -\pi < \theta < \pi \quad \text{A. 529}$$

$$= \frac{\theta^2}{\sin 2\theta} \quad \text{L. 82}$$

$$= e^{k\theta} \quad \text{where } 0 < \theta < \pi$$

$$= e^{k\theta} \quad \text{where } 0 < \theta < \pi$$

$$= \frac{nx^{n-1}(x^n - a^n \cos \theta)}{x^{2n} - 2x^na^n \cos \theta + a^{2n}} \quad \text{Y. 55}$$

$$= \tan^{-1}(x + h) \quad \text{where } x = \cot \theta \quad \text{Y. 108}$$

$$= \frac{gd^{-1}\theta}{\sin \theta} \quad \text{Y. 505}$$

Series No.

$$(674) \quad 1 - \frac{n(n-1)}{2!} \tan^2 \theta + \frac{n(n-1)(n-2)(n-3)}{4!} \tan^4 \theta + \dots \infty$$

$$(675) \quad \sum_{-\infty}^{+\infty} \left\{ \tan^{-1} \frac{y}{n+x} - \frac{y}{n} \right\}$$

$$(676) \quad \sum_{-\infty}^{+\infty} (-1)^n \left\{ \tan^{-1} \frac{y}{n+x} - \frac{y}{n} \right\}$$

$$(677) \quad \cos \theta \cos a + \frac{1}{2} \cos 2\theta \cos 2a + \frac{1}{3} \cos 3\theta \cos 3a + \dots \infty$$

$$(678) \quad \cos \theta \cos a - \frac{1}{2} \cos 2\theta \cos 2a + \frac{1}{3} \cos 3\theta \cos 3a - \dots \infty$$

$$(679) \quad \sin \theta \sin a + \frac{1}{2} \sin 2\theta \sin 2a + \frac{1}{3} \sin 3\theta \sin 3a + \dots \infty$$

$$(680) \quad \sin \theta \sin a - \frac{1}{2} \sin 2\theta \sin 2a + \frac{1}{3} \sin 3\theta \sin 3a - \dots \infty$$

$$(681) \quad \cos \theta \cos a + \frac{1}{2^2} \cos 2\theta \cos 2a + \frac{1}{3^2} \cos 3\theta \cos 3a + \dots \infty$$

$$(682) \quad \cos \theta \cos a - \frac{1}{2^2} \cos 2\theta \cos 2a + \frac{1}{3^2} \cos 3\theta \cos 3a - \dots \infty$$

$$(683) \quad \sin \theta \sin a + \frac{1}{2^2} \sin 2\theta \sin 2a + \frac{1}{3^2} \sin 3\theta \sin 3a + \dots \infty$$

$$= \frac{\cos n\theta}{\cos^n \theta} \quad \text{where } n \text{ is a positive integer} \quad \text{D. 330}$$

$$= -\tan^{-1} \frac{y}{x} + \tan^{-1} \frac{\tanh \pi y}{\tan \pi x} \quad \text{omit } y = x \quad \text{A. 314}$$

$$= -\tan^{-1} \frac{y}{x} + \tan^{-1} \frac{\sinh \pi y}{\sin \pi x} \quad \text{omit } y = x \quad \text{A. 314}$$

$$= -\frac{1}{4} \log [4(\cos \theta - \cos a)^2] \quad \text{where } \theta \neq 2n\pi + a \quad \text{A. 358}$$

$$= \frac{1}{4} \log [4(\cos \theta + \cos a)^2] \quad \text{where } \theta \neq (2n+1)\pi \pm a \quad \text{A. 358}$$

$$= \frac{1}{4} \log \frac{\sin^2 \frac{1}{2}(\theta + a)}{\sin^2 \frac{1}{2}(\theta - a)} \quad \text{where } \theta \neq 2n\pi \pm a \quad \text{A. 358}$$

$$= \frac{1}{4} \log \frac{\cos^2 \frac{1}{2}(\theta - a)}{\cos^2 \frac{1}{2}(\theta + a)} \quad \text{where } \theta \neq (2n+1)\pi \pm a \quad \text{A. 358}$$

$$= \frac{1}{4} \theta^2 + \frac{1}{4} (a - \pi)^2 - \frac{1}{12} \pi^2 \quad \text{where } 0 \leq \theta \leq a \quad \text{A. 361}$$

$$= \frac{1}{4} a^2 + \frac{1}{4} (\theta - \pi)^2 - \frac{1}{12} \pi^2 \quad \text{where } a \leq \theta \leq \pi \quad \text{A. 361}$$

$$= \frac{1}{12} \pi^2 - \frac{1}{4} (a^2 + \theta^2) \quad \text{where } -(\pi - a) \leq \theta \leq (\pi - a) \quad \text{A. 361}$$

$$= \frac{1}{12} \pi^2 - \frac{1}{4} \{(a - \pi)^2 + (\theta - \pi)^2\} \quad \text{where } (\pi - a) \leq \theta \leq (\pi + a) \quad \text{A. 361}$$

$$= \frac{1}{2} \theta(\pi - a) \quad \text{where } -a \leq \theta \leq a \quad \text{A. 362}$$

$$= \frac{1}{2} a(\pi - \theta) \quad \text{where } a \leq \theta \leq (2\pi - a) \quad \text{A. 362}$$

Series No.

$$(684) \sin \theta \cos a + \frac{1}{2} \sin 2\theta \cos 2a + \frac{1}{3} \sin 3\theta \cos 3a + \dots \infty$$

$$\cos \theta \sin a + \frac{1}{2} \cos 2\theta \sin 2a + \frac{1}{3} \cos 3\theta \sin 3a + \dots \infty$$

$$(685) \cos \theta + a \cos (\theta + \beta) + a^2 \cos (\theta + 2\beta) + \dots \infty$$

$$(686) \sin \theta + a \sin (\theta + \beta) + a^2 \sin (\theta + 2\beta) + \dots \infty$$

$$(687) \cos (\theta + a) + \frac{1}{2} \cos 2(\theta + a) + \frac{1}{3} \cos 3(\theta + a) + \dots \infty$$

$$\theta \neq 2n\pi \pm a$$

$$(688) \cos (\theta - a) + \frac{1}{2} \cos 2(\theta - a) + \frac{1}{3} \cos 3(\theta - a) + \dots \infty$$

$$\theta \neq 2n\pi \pm a$$

$$(689) \sin (\theta + a) + \frac{1}{2} \sin 2(\theta + a) + \frac{1}{3} \sin 3(\theta + a) + \dots \infty$$

$$(690) \sin (\theta - a) + \frac{1}{2} \sin 2(\theta - a) + \frac{1}{3} \sin 3(\theta - a) - \dots \infty$$

$$(691) \sin \theta + a \sin (\theta + \beta) + \frac{a^2}{2!} \sin (\theta + 2\beta) + \dots \infty$$

Reference

$$= f(\theta) \quad \text{where } \left. \begin{aligned} f(\theta) &= -\frac{1}{2}\theta \\ g(\theta) &= \frac{1}{2}(\pi - a) \end{aligned} \right\} \quad \text{if } 0 < \theta < a$$

$$= g(\theta) \quad \text{where } \left. \begin{aligned} f(\theta) &= \frac{1}{2}(\pi - \theta) \\ g(\theta) &= -\frac{1}{2}a \end{aligned} \right\} \quad \text{if } a < \theta < \pi$$

$$f(a) = \frac{1}{4}(\pi - 2a) = g(a) \quad \text{A. 358}$$

$$= \frac{\cos \theta - a \cos (\theta - \beta)}{1 - 2a \cos \beta + a^2} \quad \text{E. 131}$$

$$= \frac{\sin \theta - a \sin (\theta - \beta)}{1 - 2a \cos \beta + a^2} \quad \text{E. 117}$$

$$= -\frac{1}{2} \log h \left[4 \sin^2 \frac{1}{2}(\theta + a) \right] \quad \text{A. 358}$$

$$= -\frac{1}{2} \log h \left[4 \sin^2 \frac{1}{2}(\theta - a) \right] \quad \text{A. 358}$$

$$= \frac{1}{2} \{ \pi - (\theta + a) \} \quad \text{where } 0 < a < \pi \quad \text{and} \quad 0 < (\theta + a) < 2\pi$$

A. 358

$$= \frac{1}{2} \{ \pi - (\theta - a) \} \quad \text{where } 0 < a < \pi \quad \text{and} \quad 0 < (\theta - a) < 2\pi;$$

when $\theta < a$ the sum is diminished by π A. 358

$$= e^{\alpha \cos \beta} \sin (\theta + a \sin \beta) \quad \text{E. 121}$$

Series No.

$$(692) \cos \theta + a \cos (\theta + \beta) + \frac{a^2}{2!} \cos (\theta + 2\beta) + \dots \infty$$

$$(693) 1 - \cos \theta \cos \beta + \frac{\cos^2 \theta}{2!} \cos 2\beta - \frac{\cos^3 \theta}{3!} \cos 3\beta + \dots \infty$$

$$(694) \sin \theta - \frac{1}{2!} \sin (\theta + 2\beta) + \frac{1}{4!} \sin (\theta + 4\beta) - \dots \infty$$

$$(695) \cos \theta - \frac{1}{3!} \cos (\theta + 2\beta) + \frac{1}{5!} \cos (\theta + 4\beta) - \dots \infty$$

$$(696) a \cos \theta - \frac{a^3}{3} \cos (\theta + 2\beta) + \frac{a^5}{5} \cos (\theta + 4\beta) + \dots \infty$$

$$(697) \sum_1^{\infty} \frac{1}{n^2} \sin n\theta \sin na \cos n\beta$$

$$(698) \sin \theta \cos \theta + \sin^2 \theta \cos 2\theta + \dots \infty$$

$$(699) \sum_1^{\infty} \frac{1}{n} \sin 2n\theta \sin^2 n\phi$$

$$(700) \sum_1^{\infty} \frac{1}{n^2} \sin^2 n\theta \sin^2 n\phi$$

Reference

$$= e^{a \cos \beta} \cos (\theta + a \sin \beta) \quad \text{E. 121}$$

$$= e^{-\cos \theta \cos \beta} \cos (\cos \theta \sin \beta) \quad \text{E. 121}$$

$$\doteq \sin \theta \cos (\cos \beta) \cosh (\sin \beta) - \cos \theta \sin (\cos \beta) \sinh (\sin \beta) \quad \text{E. 121}$$

$$\begin{aligned} &= \sin (\cos \beta) \cosh (\sin \beta) \cos (\theta - \beta) \\ &\quad - \cos (\cos \beta) \sinh (\sin \beta) \sin (\theta - \beta) \quad \text{E. 122} \end{aligned}$$

$$= \frac{1}{2} \cos (\theta - \beta) \tan^{-1} \frac{2a \cos \beta}{1 - a^2} - \frac{1}{2} \sin (\theta - \beta) \tanh^{-1} \frac{2a \sin \beta}{1 + a^2} \quad \text{E. 122}$$

$$\text{where } 0 < \beta < a < \frac{\pi}{2}$$

$$= \frac{1}{2} \theta (\pi - a) \quad \text{where } 0 < \theta < (a - \beta)$$

$$= \frac{1}{4} \pi (\theta + a - \beta) - \frac{1}{2} a \theta \quad \text{where } (a - \beta) < \theta < (a + \beta)$$

$$= \frac{1}{2} a (\pi - \theta) \quad \text{where } (a + \beta) < \theta < \pi \quad \text{A. 390}$$

$$= \frac{\sin \theta (\cos \theta - \sin \theta)}{1 - \sin 2\theta + \sin^2 \theta} \quad \text{where } \theta \neq \pm \frac{\pi}{2} \quad \text{E. 117}$$

$$= \frac{1}{4} \pi \quad \text{where } \begin{matrix} 0 < 2\theta < \pi \\ \theta < \phi < (\pi - \theta) \end{matrix} \quad \text{A. 391}$$

$$= \frac{1}{4} \pi \theta \quad \text{A. 391}$$

Series No.

$$(701) \sum_1^{\infty} \frac{1}{n^2} \sin^4 n\theta \sin^2 n\phi$$

$$(702) \sum_1^{\infty} \frac{1}{n^4} \sin^4 n\theta \sin^2 n\phi$$

$$(703) \cos \theta \sin \theta + \frac{1}{2} \cos^2 \theta \sin 2\theta + \frac{1}{3} \cos^3 \theta \sin 3\theta + \dots \infty$$

$$(704) \sec^2 \theta - \frac{1}{3} \tan^2 \theta \sec^2 \theta + \frac{1}{5} \tan^4 \theta \sec^2 \theta + \dots \infty$$

$$(705) \cot \theta \operatorname{cosec}^2 \theta - \cot 3\theta \operatorname{cosec}^2 3\theta \\ + \cot 5\theta \operatorname{cosec}^2 5\theta + \dots n \text{ terms}$$

$$(706) \theta + \cos \theta \sin \theta + \frac{\cos^2 \theta}{2} \sin 2\theta + \frac{\cos^3 \theta}{3} \sin 3\theta + \dots \infty$$

$$(707) \frac{\sin \theta}{\cos \theta} + \frac{1}{2} \frac{\sin 2\theta}{\cos^2 \theta} + \frac{1}{3} \frac{\sin 3\theta}{\cos^3 \theta} + \dots \infty$$

$$(708) \log \sin \theta + a \cot \theta - \frac{a^2}{2} \operatorname{cosec}^2 \theta + \frac{a^3}{3} \frac{\cos \theta}{\sin^3 \theta} - \dots \infty$$

$$(709) \frac{1}{n} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \tan \frac{m\pi}{n} \cos 2m\pi\theta$$

$$(710) \frac{1}{3} + 4\pi^{-1} \sum_{m=1}^{\infty} m^{-1} \sin \frac{2}{3} m\pi \cos 2m\pi\theta$$

Reference

$$= \frac{1}{8} \pi \theta \quad \text{A. 391}$$

$$= \frac{1}{6} \pi \theta^3 \quad \text{A. 391}$$

$$= \frac{\pi}{2} - \theta \quad \text{where } 0 < \theta < \frac{\pi}{2} \quad \text{E. 121}$$

$$= \frac{\theta}{\sin \theta \cos \theta} \quad \text{2Z. 246}$$

$$= 2n^3 \quad \text{A. 223}$$

$$= \frac{\pi}{2} \quad \text{Y. 108}$$

$$= \frac{\pi}{2} \quad \text{L. 81}$$

$$= \log h \sin (\theta + a) \quad \text{L. 71}$$

$$= (-1)^s \quad \text{if } \theta \text{ is not a multiple of } \frac{1}{n}$$

$$= 0 \quad \text{if } \theta \text{ is a multiple integer of } \frac{1}{n} \quad \text{Q. 191}$$

$$\begin{cases} \theta \text{ is a real variable between 0 and 1} \\ n \text{ is an odd integer } \geq 3 \\ s \text{ is the greatest integer in } n\theta \end{cases}$$

$$= 1 \quad \text{when } 0 < \theta < \frac{1}{3} \text{ and } \frac{2}{3} < \theta < 1$$

$$= -1 \quad \text{when } \frac{1}{3} < \theta < \frac{2}{3} \quad \text{Q. 191}$$

*Series No.***XV. Hyperbolic Summations**

$$(711) \quad 1 + \cosh \theta + \frac{\cosh 2\theta}{2!} + \frac{\cosh 3\theta}{3!} + \dots \infty$$

$$(712) \quad \sinh \theta + \frac{\sinh 2\theta}{2!} + \frac{\sinh 3\theta}{3!} + \dots \infty$$

$$(713) \quad 1 - \frac{2}{3} \sinh^2 \theta + \frac{2 \cdot 4}{3 \cdot 5} \sinh^4 \theta - \dots \infty$$

$$(714) \quad 1 + a \cosh \theta + a^2 \cosh 2\theta + \dots a^{n-1} \cosh (n-1)\theta$$

$$(715) \quad a \sinh \theta + a^2 \sinh 2\theta + \dots \infty$$

$$(716) \quad \sinh \theta - \frac{1}{2} \sinh 2\theta + \frac{1}{3} \sinh 3\theta - \dots \infty$$

$$(717) \quad \sinh^2 \theta - 2^2 \sinh^4 \frac{\theta}{2} - 2^4 \sinh^4 \frac{\theta}{2^2} - 2^6 \sinh^4 \frac{\theta}{2^3} - \dots \infty$$

$$(718) \quad \tanh \theta + \frac{1}{3} \tanh^3 \theta + \frac{1}{5} \tanh^5 \theta + \dots \infty$$

$$(719) \quad \coth^2 \theta - \sum_1^n \frac{1}{2^{2n}} \tanh^2 \frac{\theta}{2^n}$$

$$(720) \dagger \quad \frac{1}{n} \sum_{r=0}^{\frac{n}{2}-1} \frac{\tanh \theta \frac{2}{n \sin^2 \frac{2r+1}{2n} \pi}}{1 + \frac{\tanh^2 \theta}{\tan^2 \frac{2r+1}{2n} \pi}}$$

$$\dagger \quad \frac{\frac{2 \tanh \theta}{n \sin^2 R\pi}}{1 + \frac{\tanh^2 2\theta}{\tan^2 R\pi}} = \frac{2 \sinh 2\theta}{n(\cosh 2\theta - \cos 2R\pi)}$$

$$= e^{\cosh \theta} \cosh (\sinh \theta) \quad \text{E. 122}$$

$$= e^{\cosh \theta} \sinh (\sinh \theta) \quad \text{E. 122}$$

$$= \frac{\theta}{\sinh \theta \cosh \theta} \quad \text{A. 198}$$

$$= \frac{1 - a \cosh \theta - a^n \cosh n\theta + a^{n+1} \cosh (n-1)\theta}{1 - 2a \cosh \theta + a^2} \quad \text{E. 117}$$

$$= \frac{a \sinh \theta}{1 - 2a \cosh \theta + a^2} \quad \text{E. 117}$$

$$= \frac{\theta}{2} \quad \text{E. 122}$$

$$= \theta^2 \quad \text{C. 336}$$

$$= \tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta - \dots \infty \quad \text{where } \theta \text{ is between } \pm \frac{\pi}{4} \quad \text{E. 123}$$

$$= \frac{1}{2^{2n}} \coth^2 \frac{\theta}{2^{2n}} \quad \text{C. 315}$$

$$= \tanh n\theta \quad \text{where } n \text{ is even}$$

Series No.

$$(721)^\dagger \frac{1}{n} (\tanh \theta + \coth \theta) + \sum_{r=1}^{\frac{n}{2}-1} \frac{\tanh \theta \frac{2}{n \sin^2 \frac{r\pi}{n}}}{1 + \frac{\tanh^2 \theta}{\tan^2 \frac{r\pi}{n}}}$$

$$(722)^\dagger \frac{\tanh \theta}{n} + \sum_{r=0}^{\frac{n-3}{2}} \frac{\tanh \theta \frac{2}{n \sin^2 \frac{2r+1}{2n} \pi}}{1 + \frac{\tanh^2 \theta}{\tan^2 \frac{2r+1}{2n} \pi}}$$

$$(723)^\dagger \frac{\coth \theta}{n} + \sum_{r=1}^{\frac{n-1}{2}} \frac{\tanh \theta \frac{2}{n \sin^2 \frac{r\pi}{n}}}{1 + \frac{\tanh^2 \theta}{\tan^2 \frac{r\pi}{n}}}$$

$$(724)^\ddagger \frac{1}{n} \sum_{r=0}^{\frac{n}{2}-1} \frac{1}{\sin^2 \frac{2r+1}{2n} \pi} \frac{1}{\frac{\sinh \theta}{\cosh \theta} + \frac{1}{2} \tanh \frac{\theta}{2}}$$

$$^\dagger \frac{\frac{2 \tanh \theta}{n \sin^2 \frac{R\pi}{n}}}{1 + \frac{\tanh^2 \theta}{\tan^2 \frac{R\pi}{n}}} = \frac{2 \sinh 2\theta}{n \{ \cosh 2\theta - \cos 2R\pi \}}$$

$$^\ddagger \frac{1}{\frac{\sin^2 R\pi}{\sinh \theta} + \frac{1}{2} \tanh \frac{\theta}{2}} = \frac{2 \sinh \theta}{\cosh \theta - \cos 2R\pi}$$

$$= \coth n\theta \quad \text{where } n \text{ is even}$$

$$= \tanh n\theta \quad \text{where } n \text{ is odd}$$

$$= \coth n\theta \quad \text{where } n \text{ is odd}$$

$$= \tanh \frac{n\theta}{2} \quad \text{where } n \text{ is even}$$

Series No.

$$(725) \dagger \frac{1}{n} \left\{ 2 \coth \theta + \sum_{r=1}^{\frac{n}{2}-1} \frac{1}{\frac{\sin^2 \frac{r\pi}{n}}{\sinh \theta} + \frac{1}{2} \tanh \frac{\theta}{2}} \right\}$$

$$(726) \dagger \frac{1}{n} \left\{ \tanh \frac{\theta}{2} + \sum_{r=0}^{\frac{n-3}{2}} \frac{1}{\frac{\sin^2 \frac{2r+1}{2n} \pi}{\sinh \theta} + \frac{1}{2} \tanh \frac{\theta}{2}} \right\}$$

$$(727) \dagger \frac{1}{n} \left\{ \coth \frac{\theta}{2} + \sum_{r=1}^{\frac{n-1}{2}} \frac{1}{\frac{\sin^2 \frac{r\pi}{n}}{\sinh \theta} + \frac{1}{2} \tanh \frac{\theta}{2}} \right\}$$

$$(728) \frac{1}{a} + 2a \sum_1^{\infty} \frac{\cos n\theta}{a^2 + n^2}$$

$$(729) 2 \sum_1^{\infty} \frac{n \sin n\theta}{n^2 + a^2}$$

$$(730) \frac{1}{a} + 2a \sum_1^{\infty} \frac{(-1)^n \cos n\theta}{a^2 + n^2}$$

$$(731) 2 \sum_1^{\infty} \frac{(-1)^{n-1} n \sin n\theta}{n^2 + a^2}$$

XVI. Trigonometrical Expansions

$$(732) 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \infty = \sum_1^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!}$$

$$(733) \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \infty = \sum_0^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}$$

$$\dagger \frac{1}{\frac{\sin^2 R\pi}{\sinh \theta} + \frac{1}{2} \tanh \frac{\theta}{2}} = \frac{2 \sinh \theta}{\cosh \theta - \cos 2R\pi}$$

$$= \coth \frac{n\theta}{2} \quad \text{where } n \text{ is even}$$

$$= \tanh \frac{n\theta}{2} \quad \text{where } n \text{ is odd}$$

$$= \coth \frac{n\theta}{2} \quad \text{where } n \text{ is odd}$$

$$= \pi \frac{\cosh a(\pi - \theta)}{\sinh a\pi} \quad \text{where } 0 \leq \theta \leq 2\pi \quad \text{A. 393}$$

$$= \pi \frac{\sinh (\pi - \theta)a}{\sinh a\pi} \quad \text{where } 0 < \theta < 2\pi \quad \text{A. 393}$$

$$= \pi \frac{\cosh a\theta}{\sinh a\pi} \quad \text{where } -\pi \leq \theta \leq \pi \quad \text{A. 368}$$

$$= \pi \frac{\sinh a\theta}{\sinh a\theta} \quad \text{where } -\pi < \theta < \pi \quad \text{A. 368}$$

$$= \cos \theta \quad \text{where } \theta < \infty$$

$$= \sin \theta \quad \text{where } \theta < \infty$$

Series No.

$$(734) \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \dots \infty = \sum_0^{\infty} (-1)^n \frac{\theta^{2n+1}}{2n+1}$$

In the following series Nos. (735) through (763), see No. (1130) for values of the coefficients.

$$(735) 1 - B_1 \frac{\theta^2}{2!} - B_2 \frac{\theta^4}{4!} - \dots \infty = 1 - \sum_1^{\infty} B_n \frac{\theta^{2n}}{(2n)!}$$

$$(736) \alpha_1 \frac{\theta}{2!} + \alpha_2 \frac{\theta^3}{4!} + \dots \infty = \sum_1^{\infty} \alpha_n \frac{\theta^{2n-1}}{(2n)!}$$

$$(737) 1 + \beta_1 \frac{\theta^2}{2!} + \beta_2 \frac{\theta^4}{4!} + \dots \infty = 1 + \sum_1^{\infty} \beta_n \frac{\theta^{2n}}{(2n)!}$$

$$(738) 1 + E_2 \frac{\theta^2}{2!} + E_4 \frac{\theta^4}{4!} + \dots \infty = 1 + \sum_1^{\infty} E_{2n} \frac{\theta^{2n}}{(2n)!}$$

$$(739) 2 + \delta_1 \frac{\theta^2}{2!} + \delta_2 \frac{\theta^4}{4!} + \dots \infty = 2 + \sum_1^{\infty} \delta_n \frac{\theta^{2n}}{(2n)!}$$

$$(740) \zeta_1 \frac{\theta}{2!} + \zeta_2 \frac{\theta^3}{4!} + \zeta_3 \frac{\theta^5}{6!} + \dots \infty = \sum_1^{\infty} \zeta_n \frac{\theta^{2n-1}}{(2n)!}$$

$$(741) I_0 + I_1 \frac{\theta^2}{2!} + I_2 \frac{\theta^4}{4!} + \dots \infty = I_0 + \sum_1^{\infty} I_n \frac{\theta^{2n}}{(2n)!}$$

$$(742) H_0 + H_1 \frac{\theta^2}{2!} + H_2 \frac{\theta^4}{4!} + \dots \infty = H_0 + \sum_1^{\infty} H_n \frac{\theta^{2n}}{(2n)!}$$

$$(743) \gamma_1 \frac{\theta}{2!} + \gamma_2 \frac{\theta^3}{4!} + \gamma_3 \frac{\theta^5}{6!} + \dots \infty = \sum_1^{\infty} \gamma_n \frac{\theta^{2n-1}}{(2n)!}$$

Reference

$$= \tan^{-1} \theta \quad \text{where } \theta^2 \leq 1$$

E. 107

$$= \frac{\theta}{2} \cot \frac{\theta}{2}$$

AC. 41

$$= \tan \frac{\theta}{2}$$

AC. 41

$$= \theta \operatorname{cosec} \theta$$

AC. 41

$$= \sec \theta$$

AC. 41

$$= \frac{3\theta \cos \frac{\theta}{2}}{\sin \frac{3}{2}\theta}$$

AC. 41

$$= \frac{\sin \frac{\theta}{2}}{\cos \frac{3\theta}{2}}$$

AC. 41

$$= \frac{3 \sin \frac{\theta}{2}}{2 \sin \frac{3\theta}{2}}$$

AC. 41

$$= \frac{3 \cos \frac{\theta}{2}}{2 \cos \frac{3\theta}{2}}$$

AC. 41

$$= \frac{3 \sin \frac{\theta}{2} \sin \theta}{2 \sin \frac{3\theta}{2}}$$

AC. 41

Series No.

$$(744) \theta_1 \frac{\theta}{2!} + \theta_2 \frac{\theta^3}{4!} + \theta_3 \frac{\theta^5}{6!} + \dots \infty = \sum_1^{\infty} \theta_n \frac{\theta^{2n-1}}{(2n)!}$$

$$(745) 1 - \epsilon_1 \frac{\theta^2}{2!} - \epsilon_2 \frac{\theta^4}{4!} - \dots \infty = 1 - \sum_1^{\infty} \epsilon_n \frac{\theta^{2n}}{(2n)!}$$

$$(746) T_1 \theta + T_2 \frac{\theta^3}{3!} + T_3 \frac{\theta^5}{5!} + \dots \infty = \sum_1^{\infty} T_n \frac{\theta^{2n-1}}{(2n-1)!}$$

$$(747) J_0 + J_1 \frac{\theta^2}{2!} + J_2 \frac{\theta^4}{4!} + \dots \infty = J_0 + \sum_1^{\infty} J_n \frac{\theta^{2n}}{(2n)!}$$

$$(748) S_0 + S_1 \frac{\theta^2}{2!} + S_2 \frac{\theta^4}{4!} + \dots \infty = S_0 + \sum_1^{\infty} S_n \frac{\theta^{2n}}{(2n)!}$$

$$(749) \eta_1 \frac{\theta}{2!} + \eta_2 \frac{\theta^3}{4!} + \eta_3 \frac{\theta^5}{6!} + \dots \infty = \sum_1^{\infty} \eta_n \frac{\theta^{2n-1}}{(2n)!}$$

$$(750) R_0 + R_1 \frac{\theta^2}{2!} + R_2 \frac{\theta^4}{4!} + \dots \infty = R_0 + \sum_1^{\infty} R_n \frac{\theta^{2n}}{(2n)!}$$

$$(751) P_0 + P_1 \frac{\theta^2}{2!} + P_2 \frac{\theta^4}{4!} + \dots \infty = P_0 + \sum_1^{\infty} P_n \frac{\theta^{2n}}{(2n)!}$$

$$(752) Q_1 \theta + Q_2 \frac{\theta^3}{2!} + Q_3 \frac{\theta^5}{4!} + \dots \infty = \sum_1^{\infty} Q_n \frac{\theta^{2n-1}}{(2n-2)!}$$

$$(753) 1 + E_1 \theta + E_2 \frac{\theta^2}{2!} + E_3 \frac{\theta^3}{3!} + \dots \infty$$

$$(754) \theta + E_2 \frac{\theta^3}{3!} + E_4 \frac{\theta^5}{5!} + \dots \infty$$

$$(755) \log \theta + \sum_1^{\infty} \frac{(2^{2n-1} - 1)2^{2n}}{n \cdot (2n)!} B_n \theta^{2n}$$

$$(756) B_1 \theta^2 + B_2 \frac{2^3}{2 \cdot 4!} \theta^4 + B_3 \frac{2^5}{3 \cdot 6!} \theta^6 + \dots \infty$$

Reference

$= \frac{\sin \theta \cos \frac{\theta}{2}}{2 \cos \frac{3\theta}{2}}$	AC. 41
$= \frac{3\theta \cos 2\theta}{\sin 3\theta}$	AC. 41
$= \frac{\sin 2\theta}{2 \cos 3\theta}$	AC. 41
$= \frac{3 \sin 2\theta}{\sin 3\theta}$	AC. 41
$= \frac{\cos 2\theta}{\cos 3\theta}$	AC. 41
$= \frac{3 \sin^2 \theta}{2 \sin 3\theta}$	AC. 41
$= \frac{\cos^2 \theta}{\cos 3\theta}$	AC. 41
$= \frac{\cos \theta}{\cos 2\theta}$	AC. 41
$= \frac{\sin \theta}{\cos 2\theta}$	AC. 41
$= \sec \theta + \tan \theta$	Y. 500
$= \log h (\sec \theta + \tan \theta) = g d^{-1} \theta$	Y. 500
$= \log h \tan \theta \quad \text{where } \theta^2 < \frac{\pi^2}{4}$	T. 123
$= -\log h \frac{\sin \theta}{\theta}$	B. 245

Series No.

$$(757) \sum_1^{\infty} \log h \left\{ 1 - \frac{4\theta^2}{(2n-1)^2\pi^2} \right\} = - \sum_1^{\infty} \frac{2^{2n-1}(2^n-1)}{n(2n)!} B_n \theta^{2n}$$

$$(758) \log h \theta + \sum_1^{\infty} \log h \left\{ 1 - \frac{\theta^2}{n^2\pi^2} \right\} = \log h \theta - \sum_1^{\infty} \frac{2^{2n-1}}{n(2n)!} B_n \theta^{2n}$$

$$(759) \frac{B_1}{2} \cdot \frac{\theta^2}{2!} + \frac{B_2}{4} \cdot \frac{\theta^4}{4!} + \frac{B_3}{6} \cdot \frac{\theta^6}{6!} + \dots \infty$$

$$(760) - \sum_1^{\infty} 2^{n+1} \cos \frac{n\pi}{2} \frac{B_n}{2!} \frac{\theta^{2n}}{(2n)!}$$

$$(761) \frac{1}{\theta} + \frac{\theta}{3!} + \frac{7}{3 \cdot 5!} \theta^3 + \dots \infty$$

$$= \frac{1}{\theta} + \sum_0^{\infty} \frac{2(2^{2n+1}-1)}{(2n+2)!} B_{n+1} \theta^{2n+1}$$

$$(762) \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{45} - \dots \infty = \frac{1}{\theta} - \sum_1^{\infty} \frac{2^{2n}}{(2n)!} B_n \theta^{2n-1}$$

$$(763) \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \dots \infty = \sum_1^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \theta^{2n-1}$$

$$(764) \sum_0^{\infty} \frac{8\theta}{(2n+1)^2\pi^2 - 4\theta^2} \theta$$

$$(765) \frac{2}{\pi - 2\theta} - \frac{2}{\pi + 2\theta} + \frac{2}{3\pi - 2\theta} - \frac{2}{3\pi + 2\theta} + \dots \infty$$

$$(766) 2 \sum_1^{\infty} \frac{\theta}{\left(\frac{2n-1}{2}\right)^2 \pi^2 - \theta^2}$$

$$(767) \frac{1}{\theta} + \frac{1}{\theta - \pi} + \frac{1}{\theta + \pi} + \frac{1}{\theta - 2\pi} + \frac{1}{\theta + 2\pi} + \dots \infty$$

$$= \frac{1}{\theta} - \sum_1^{\infty} \frac{2\theta}{n^2\pi^2 - \theta^2}$$

$$(768) \frac{4}{\pi - 2\theta} - \frac{4}{3\pi + 2\theta} + \frac{4}{5\pi - 2\theta} - \frac{4}{7\pi + 2\theta} + \dots \infty$$

Reference

$$= \log \cos \theta \quad \text{where } \theta^2 < \frac{\pi^2}{4} \quad \text{B. 237}$$

$$= \log \sin \theta \quad \text{where } \theta^2 < \pi^2 \quad \text{B. 237}$$

$$= \log \frac{\theta}{2 \sin \frac{\theta}{2}} \quad \text{A. 315}$$

$$= \log \frac{\cosh \theta - \cos \theta}{\theta^2} \quad \text{A. 315}$$

$$= \operatorname{cosec} \theta \quad \text{where } \theta^2 < \pi^2 \quad \text{T. 121}$$

$$= \cot \theta \quad \text{where } \theta^2 < \pi^2 \quad \text{T. 121}$$

$$= \tan \theta \quad \text{where } \theta^2 < \frac{\pi^2}{4} \quad \text{T. 121}$$

$$= \tan \theta \quad \text{where } \theta \neq n\pi \quad \text{A. 296}$$

$$= \tan \theta \quad \text{where } \theta \neq \frac{(2n+1)\pi}{2} \quad \text{C. 360}$$

$$= \tan \theta \quad \text{B. 237}$$

$$= \cot \theta \quad \text{where } \theta \neq n\pi \quad \text{C. 360}$$

$$= \sec \theta + \tan \theta \quad \text{where } \frac{\pi}{2} > \theta > -\frac{\pi}{2} \quad \text{Y. 501}$$

Series No.

$$(769) \frac{1}{\theta} + \frac{1}{\pi - \theta} - \frac{1}{\pi + \theta} - \frac{1}{2\pi - \theta} + \frac{1}{2\pi + \theta} + \dots \infty$$

$$= \frac{1}{\theta} + 2\theta \sum_1^{\infty} \frac{(-1)^{n-1}}{n^2\pi^2 - \theta^2}$$

$$(770) \frac{2}{\pi - 2\theta} + \frac{2}{\pi + 2\theta} - \frac{2}{3\pi - 2\theta} - \frac{2}{3\pi + 2\theta} - \dots \infty$$

$$= \sum_1^{\infty} (-1)^{n-1} \frac{(2n-1)\pi}{\left(\frac{2n-1}{2}\right)^2 \pi^2 - \theta^2}$$

$$(771) \frac{1}{\pi - 4\theta^2} - \frac{3}{3^2\pi^2 - 4\theta^2} + \frac{5}{5^2\pi^2 - 4\theta^2} - \dots \infty$$

$$(772) \frac{1}{\theta^2} - \frac{1}{\theta^2 - \pi^2} + \frac{1}{\theta^2 - 2^2\pi^2} - \dots \infty$$

$$(773) \frac{1}{(\pi - 2\theta)^2} + \frac{1}{(\pi + 2\theta)^2} + \frac{1}{(3\pi - 2\theta)^2} + \frac{1}{(3\pi + 2\theta)^2} + \dots \infty$$

$$(774) 1 + 2\theta + 2\theta^2 + \frac{8\theta^3}{3} + \frac{10\theta^4}{3} + \dots \infty$$

$$(775) \theta + \frac{1}{2} \cdot \frac{\theta^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\theta^5}{5} + \dots \infty = \sum_0^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} \theta^{2n+1}$$

$$(776) 1 + \frac{\theta^2}{3!} + \frac{14\theta^4}{6!} + \dots \infty$$

$$(777) 1 - \theta^2 + \frac{2^3\theta^4}{4!} - \frac{2^5\theta^6}{6!} + \dots \infty$$

$$(778) \theta^2 + \frac{2}{3} \cdot \frac{\theta^4}{2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{\theta^6}{3} + \dots \infty = \sum_0^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!(n+1)} \theta^{2n+2}$$

$$(779) 1 + \frac{1}{3 \cdot 4} \theta + \frac{1 \cdot 3}{5 \cdot 4^2} \frac{\theta^2}{2!} + \dots \infty$$

$$= 1 + \sum_1^{\infty} \frac{1 \cdot 3 \dots (2n-1)}{(2n+1)4^n} \cdot \frac{\theta^n}{n!}$$

$$(780) \theta + \frac{1}{3} \cdot \frac{\theta^2}{2} + \frac{1 \cdot 2}{3 \cdot 5} \frac{\theta^3}{3} + \dots \infty = \theta + \sum_2^{\infty} \frac{(n-1)!}{3 \cdot 5 \dots 2n-1} \frac{\theta^n}{n}$$

$$= \operatorname{cosec} \theta \quad \text{where } \theta \neq n\pi \quad \text{C. 361}$$

$$= \sec \theta \quad \text{where } \theta \neq \frac{2n \pm 1}{2} \pi \quad \text{C. 361}$$

$$= \frac{\sec \theta}{4\pi} \quad \text{E. 158}$$

$$= \frac{1 + \theta \operatorname{cosec} \theta}{2\theta^2} \quad \text{E. 158}$$

$$= \frac{\sec^2 \theta}{4} \quad \text{E. 159}$$

$$= \tan \left\{ \frac{\pi}{4} + \theta \right\} \quad \text{H. 498}$$

$$= \sin^{-1} \theta = \frac{\pi}{2} - \cos^{-1} \theta \quad \text{where } \theta^2 \leq 1 \quad \text{T. 121}$$

$$= \frac{\theta}{\sin \theta} \quad \text{H. 498}$$

$$= \cos^2 \theta \quad \text{H. 498}$$

$$= (\sin^{-1} \theta)^2 \quad \text{where } |\theta^2| < | \quad \text{T. 122}$$

$$= \frac{\operatorname{vers}^{-1} \theta}{\sqrt{2\theta}} \quad \text{Y. 505}$$

$$= \frac{(\operatorname{vers}^{-1} \theta)^2}{2} \quad \text{Y. 505}$$

Series No.

$$(781) \frac{\theta^5}{1 \cdot 3 \cdot 5} - \frac{\theta^7}{3 \cdot 5 \cdot 7} + \frac{\theta^9}{5 \cdot 7 \cdot 9} - \dots \infty$$

$$(782) \theta^2 - \frac{\theta^4}{3} + \frac{2\theta^6}{45} - \dots \infty$$

$$(783) \theta + \frac{2^2}{3!} \theta^3 + \frac{2^2 \cdot 4^2}{5!} \theta^5 + \frac{2^2 \cdot 4^2 \cdot 6^2}{7!} \theta^7 + \dots \infty$$

$$(784) \theta - \frac{\theta^3}{3} - \frac{2}{3} \cdot \frac{\theta^5}{5} - \frac{2 \cdot 4}{3 \cdot 5} \frac{\theta^7}{7} - \dots \infty$$

$$(785) \frac{\theta^3}{1 \cdot 3} - \frac{\theta^5}{3 \cdot 5} + \frac{\theta^7}{5 \cdot 7} - \dots \infty$$

$$(786) 1 + \theta - \frac{2\theta^3}{3!} - \frac{2^2\theta^4}{4!} - \frac{2^2\theta^5}{5!} + \frac{2^3\theta^7}{7!} - \dots \infty$$

$$(787) \theta + \theta^2 + \frac{2\theta^3}{3!} - \frac{2^2\theta^5}{5!} - \frac{2^3\theta^6}{6!} - \frac{2^3\theta^7}{7!} - \dots \infty$$

$$(788) \frac{\theta^2}{3} + \frac{7\theta^4}{90} + \frac{17\theta^6}{30} + \dots \infty$$

$$(789) 2\theta + \frac{4}{3} \theta^3 + \frac{4}{3} \theta^5 + \dots \infty$$

$$(790) \theta - \frac{\theta^2}{2} + \frac{\theta^3}{6} - \frac{\theta^4}{12} + \frac{\theta^5}{24} - \dots \infty$$

$$(791) -\frac{\theta^2}{3} + \frac{13}{90} \theta^4 - \frac{251}{5 \cdot 7 \cdot 9^2} \theta^6 + \dots \infty$$

$$(793) \dagger \theta^2 - \frac{2}{3} \theta^4 + \frac{61}{120} \theta^6 - \dots \infty$$

$$(794) -\frac{\theta^2}{3} - \frac{7}{90} \theta^4 - \frac{62}{2835} \theta^6 - \dots \infty$$

$$(795) \theta - \frac{\theta^2}{2} + \frac{2\theta^3}{3} - \dots \infty$$

$$(796) \frac{\theta^2}{2} + \frac{\theta^4}{12} + \frac{\theta^6}{45} + \dots \infty$$

† No. (792) has been omitted because it duplicates a previous series.—Ed.

Reference

$= \frac{(\theta^2 + 1)^2}{8} \tan^{-1} \theta - \frac{5\theta^3 + 3\theta}{24}$	1Z. 135
$= \sin^2 \theta$	D. 336
$= \frac{\sin^{-1} \theta}{\sqrt{1 - \theta^2}} \text{ where } \theta < 1$	L. 78
$= \sqrt{1 - \theta^2} \sin^{-1} \theta$	A. 191
$= \frac{1}{2} (1 + \theta^2) \tan^{-1} \theta - \frac{1}{2} \theta$	H. 475
$= \epsilon^\theta \cos \theta$	H. 497
$= \epsilon^\theta \sin \theta$	H. 497
$= \log h \frac{\tan \theta}{\theta}$	L. 80
$= \log h \tan \left\{ \frac{\pi}{4} + \theta \right\}$	H. 498
$= \log h (1 + \sin \theta)$	H. 498
$= \log h \frac{\tan^{-1} \theta}{\theta}$	Y. 80
$= \log h (1 + \theta \sin \theta)$	Y. 106
$= \log h \theta \cot \theta$	Y. 80
$= \log h (1 + \tan \theta)$	L. 79
$= \log h \sec \theta$	H. 497

Series No.

$$(797) 1 + \theta + \frac{\theta^2}{2} - \frac{\theta^4}{8} - \frac{\theta^5}{15} + \frac{\theta^6}{240} - \dots \infty$$

$$(798) \epsilon \left\{ 1 - \frac{\theta^2}{2!} + \frac{4\theta^4}{4!} - \frac{31\theta^6}{6!} + \dots \infty \right\}$$

$$(799) 1 + \theta^2 + \frac{1}{3}\theta^4 + \frac{1}{120}\theta^6 + \dots \infty$$

$$(800) -\frac{\theta^2}{2!} - \frac{2\theta^4}{4!} - \frac{16\theta^6}{6!} - \frac{272\theta^8}{8!} - \dots \infty$$

$$(801) 1 + \theta + \frac{\theta^2}{2!} + \frac{2\theta^3}{3!} + \frac{5\theta^4}{4!} + \dots \infty$$

$$(802) 1 + \theta + \frac{\theta^2}{2} - \frac{\theta^3}{3} - \frac{11\theta^4}{24} - \frac{\theta^5}{5} - \dots \infty$$

$$(803) 1 + \frac{\theta^3}{3!} + \frac{\theta^6}{6!} + \frac{\theta^9}{9!} + \dots \infty$$

$$(804) \theta + \frac{\theta^4}{4!} + \frac{\theta^7}{7!} + \frac{\theta^{10}}{10!} + \dots \infty$$

$$(805) \frac{\pi}{2} - \frac{1}{\theta} + \frac{1}{3\theta^3} - \frac{1}{5\theta^5} + \frac{1}{7\theta^7} - \dots \infty$$

$$= \frac{\pi}{2} - \sum_0^{\infty} (-1)^n \frac{1}{(2n+1)\theta^{2n+1}}$$

$$(806) \frac{\pi}{2} - \frac{1}{\theta} - \frac{1}{2 \cdot 3} \frac{1}{\theta^3} - \dots \infty = \frac{\pi}{2} - \sum_0^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} \theta^{-2n-1}$$

$$(807) 1 + \theta + \frac{\theta^2}{2!} + \frac{3\theta^3}{3!} + \frac{9\theta^4}{4!} + \frac{37\theta^5}{5!} + \dots \infty$$

$$(808) 1 + \theta + \frac{\theta^2}{2} - \frac{\theta^3}{6} + \frac{7\theta^4}{24} - \dots \infty$$

$$(809) 1 + \theta + \frac{\theta^2}{2} + \frac{2\theta^3}{3} + \dots \infty$$

$$(810) \frac{2\theta}{1^2 - \theta^2} + \frac{2\theta}{3^2 - \theta^2} + \frac{2\theta}{5^2 - \theta^2} + \dots \infty$$

76229

Reference

$= e^{\sin \theta}$	L. 73
$= e^{\cos \theta}$	T. 126
$= e^{\theta \sin \theta}$	Y. 80
$= \log \cos \theta$	Y. 107
$= e^{\sin^{-1} \theta}$	T. 126
$= e^{\theta \cos \theta}$	L. 79
$= \frac{1}{3} \left\{ e^{\theta} + 2e^{-\theta/2} \cos \frac{\theta\sqrt{3}}{2} \right\}$	E. 190
$= \frac{1}{3} e^{\theta} - \frac{1}{3} e^{-\theta/2} \left\{ \cos \frac{\theta\sqrt{3}}{2} - \sqrt{3} \sin \frac{\theta\sqrt{3}}{2} \right\}$	E. 190
$= \tan^{-1} \theta \quad \text{where } \theta^2 \geq 1$	T. 122
$= \sec^{-1} \theta = \frac{\pi}{2} - \operatorname{cosec}^{-1} \theta \quad \text{where } \theta > 1$	T. 122
$= e^{\tan \theta}$	T. 126
$= e^{\tan^{-1} \theta}$	T. 126
$= e^{\theta \sec \theta}$	Y. 107
$= \frac{\pi}{2} \tan \left\{ \frac{\pi\theta}{2} \right\}$	A. 225

Series No.

$$(811) \tan^{-1} \theta + \frac{h}{1 + \theta^2} - \frac{\theta h^2}{(1 + \theta^2)^2} - \frac{(1 - 3\theta^2)h^3}{3(1 + \theta^2)^3} + \dots \infty$$

$$(812) \sin^{-1} \theta + \frac{h}{\sqrt{1 - \theta^2}} + \frac{\theta}{(1 - \theta^2)^{3/2}} \frac{h^2}{2!} + \frac{1 + 2\theta^2}{(1 - \theta^2)^{5/2}} \frac{h^3}{3!} + \dots \infty$$

$$(813) \sec^{-1} \theta + \frac{h}{\theta \sqrt{\theta^2 - 1}} - \frac{2\theta^2 - 1}{\theta^2(\theta^2 - 1)^{3/2}} \frac{h^2}{2!} + \dots \infty$$

$$(814) \frac{\theta}{1 + \theta^2} \sum_0^{\infty} \frac{2^{2n}(n!)^2}{(2n + 1)!} \left(\frac{\theta^2}{1 + \theta^2} \right)^n$$

$$(815) \frac{1}{\theta} + \sum_{-\infty}^{+\infty} \frac{\theta}{n(\theta - n)} = \frac{1}{\theta} + \sum_1^{\infty} \frac{2\theta}{\theta^2 - n^2}$$

$$(816) \frac{1}{n^2 - 1} - \frac{1}{3^3 n^2 - 3} + \frac{1}{5^3 n^2 - 5} - \frac{1}{7^3 n^2 - 7} + \dots \infty$$

$$(817) 1 - \frac{n^2}{1^2} + \frac{n^2(n^2 - 1)}{1^2 \cdot 2^2} - \frac{n^2(n^2 - 1^2)(n^2 - 2^2)}{1^2 \cdot 2^2 \cdot 3^2} + \dots \infty$$

$$(818) \left(1 + \frac{1}{1}\right) - \frac{\theta^2}{2!} \left(1 - \frac{1}{3}\right) + \frac{\theta^4}{4!} \left(1 + \frac{1}{5}\right) - \frac{\theta^6}{6!} \left(1 - \frac{1}{7}\right) + \dots \infty$$

$$(819) \frac{1}{4} \left[(1 + 3) - (3^2 + 3) \frac{\theta^2}{2} + \dots + (-1)^n \{3^{2n} + 3\} \frac{\theta^{2n}}{(2n)!} + \dots \infty \right]$$

$$(820) \frac{\theta^2}{2} - \left(1 + \frac{1}{3}\right) \frac{\theta^4}{4} + \left(1 + \frac{1}{3} + \frac{1}{5}\right) \frac{\theta^6}{6} - \dots \infty$$

$$(821) \frac{1}{2} \cdot \frac{\theta^3}{3} - \left\{ \frac{1}{2} + \frac{1}{4} \left(1 + \frac{1}{3}\right) \right\} \frac{\theta^5}{5} + \left\{ \frac{1}{2} + \frac{1}{4} \left(1 + \frac{1}{3}\right) + \frac{1}{6} \left(1 + \frac{1}{3} + \frac{1}{5}\right) \right\} \frac{\theta^7}{7} - \dots \infty$$

$$(822) \sum_{-\infty}^{+\infty} \frac{1}{(\theta - n)^2} = \frac{1}{\theta^2} + 2 \sum_1^{\infty} \frac{\theta^2 + n^2}{(\theta^2 - n^2)^2}$$

Reference

$$= \tan^{-1} (\theta + h) \quad \text{L. 71}$$

$$= \sin^{-1} (\theta + h) \quad \text{L. 71}$$

$$= \sec^{-1} (\theta + h) \quad \text{L. 71}$$

$$= \tan^{-1} \theta \quad \text{where } \theta^2 < \infty \quad \text{T. 122}$$

$$= \pi \cot \pi \theta \quad (n = 0 \text{ excluded}) \quad \text{A. 217}$$

$$= \frac{\pi}{4} \left(\sec \frac{\pi}{2n} - 1 \right) \quad \text{A. 225}$$

$$= \frac{\sin n\pi}{n\pi} \quad \text{C. 421}$$

$$= \cos \theta + \frac{\sinh \theta}{\theta} \quad \text{A. 389}$$

$$= \cos^3 \theta \quad \text{Y. 79}$$

$$= \frac{1}{2} (\tan^{-1} \theta)^2 \quad \text{where } |x| < 1 \quad \text{A. 191}$$

$$= \frac{(\tan^{-1} \theta)^3}{3!} \quad \text{Y. 89}$$

$$= \pi^2 \operatorname{cosec}^2 \pi \theta \quad \text{A. 218}$$

Series No.

$$(823) \sum_{-\infty}^{+\infty} \frac{1}{(\theta - n)^3} = \frac{1}{\theta^3} + 2 \sum_1^{\infty} \frac{\theta(\theta^2 + 3n^2)}{(\theta^2 - n^2)^3}$$

$$(824) \sum_{-\infty}^{+\infty} \frac{1}{(\theta - n)^4} = \frac{1}{\theta^4} + 2 \sum_1^{\infty} \frac{\theta^4 + 6\theta^2 n^2 + n^4}{(\theta^2 - n^2)^4}$$

$$(825) \frac{1^4 - (1^2 - \theta^2)^2}{(1^2 - \theta^2)^2} + \frac{3^4 - (3^2 - \theta^2)^2}{(1^2 - \theta^2)^2(3^2 - \theta^2)^2} + \dots \infty$$

$$(826) \frac{1}{\theta^2} + \frac{1}{3} + \frac{1}{15} \theta^2 + \frac{2}{189} \theta^4 + \dots \infty$$

$$(827) \sum_{-\infty}^{+\infty} \frac{a - \theta}{(\theta - n)(a - n)}$$

$$(828) 1 - \frac{n\theta^2}{3!} + \frac{n(5n - 2)}{3 \cdot 5!} \theta^4 + \dots \infty$$

$$(829) \frac{1}{4} \left\{ (3^3 - 3) \frac{\theta^3}{3!} + \dots (-1)^n \frac{3^{2n-1} - 3}{(2n - 1)!} \theta^{2n-1} - \dots \infty \right\}$$

$$(830) \frac{1}{\theta} + \frac{1}{\theta - 2} + \frac{1}{\theta + 2} + \frac{1}{\theta - 4} + \frac{1}{\theta + 4} + \dots \infty$$

$$(831) - \left(\frac{1}{\theta - 1} + \frac{1}{\theta + 1} \right) - \left(\frac{1}{\theta - 3} + \frac{1}{\theta + 3} \right) - \dots \infty$$

$$(832) \frac{1}{2} \frac{\theta^3}{3} + \left(\frac{1}{1^2} + \frac{1}{3^2} \right) \frac{1 \cdot 3}{2 \cdot 4} \frac{\theta^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \right) \frac{\theta^7}{7} + \dots \infty$$

$$(833) 1 + a\theta + \frac{(a^2 + 1^2)\theta^2}{2!} + \frac{a(a^2 + 2^2)\theta^3}{3!} + \dots \infty$$

$$(834) \theta + \frac{2}{3} \theta^3 + \frac{2 \cdot 4}{3 \cdot 5} \theta^5 + \dots \infty$$

$$(835) \frac{mn}{m^2 + n^2} \left\{ 1 + \frac{2}{3} \cdot \frac{m^2}{m^2 + n^2} + \frac{2 \cdot 4}{3 \cdot 5} \left(\frac{m^2}{m^2 + n^2} \right)^2 + \dots \infty \right\}$$

Reference

$$= \pi^3 \cot \pi \theta \operatorname{cosec}^2 \pi \theta \quad \text{A. 225}$$

$$= \pi^4 \left\{ \operatorname{cosec}^4 \pi \theta - \frac{2}{3} \operatorname{cosec}^2 \pi \theta \right\} \quad \text{A. 225}$$

$$= \tan^2 \frac{\pi \theta}{2} \quad \text{C. 421}$$

$$= \operatorname{cosec}^2 \theta \quad \text{A. 222}$$

$$= \pi (\cot \pi \theta - \cot \pi a) \quad \text{A. 225}$$

$$= \left(\frac{\sin \theta}{\theta} \right)^n \quad \text{H. 498}$$

$$= \sin^3 \theta \quad \text{Y. 79}$$

$$= \frac{\pi}{2} \cot \frac{\pi \theta}{2} \quad \text{A. 225}$$

$$= \frac{\pi}{2} \tan \frac{\pi \theta}{2} \quad \text{A. 225}$$

$$= \frac{1}{6} (\sin^{-1} \theta)^3 \quad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \text{A. 223}$$

$$= \frac{e^{a \sin^{-1} \theta}}{\sqrt{1 - \theta^2}} \quad \text{L. 79}$$

$$= \frac{1}{\sqrt{1 - \theta^2}} \sin^{-1} \theta \quad \text{where } |\theta| < 1 \quad \text{A. 197}$$

$$= \tan^{-1} \frac{m}{n} \quad \text{A. 196}$$

Series No.

$$(836) \theta(1 + \theta^2) - \frac{4}{1} \cdot \frac{\theta^3(1 + \theta^2)^3}{3} + \frac{6 \cdot 7}{1 \cdot 2} \frac{\theta^5(1 + \theta^2)^5}{5} \\ - \frac{8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3} \cdot \frac{\theta^7(1 + \theta^2)^7}{7} + \dots \infty$$

$$(837) 1 - \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)(n-3)}{4!} - \dots \infty$$

$$(838) (2^2 - 1) \frac{2}{2!} B_1 \theta^2 + \frac{1}{2} (2^4 - 1) \frac{2^3}{4!} B_2 \theta^4 \\ + \frac{1}{3} (2^6 - 1) \frac{2^5}{6!} B_3 \theta^6 + \dots \infty$$

$$(839) 1 + a\theta + \frac{a^2\theta^2}{2!} + \frac{a(a^2 + 1)}{3!} \theta^3 \\ + \frac{a^2(a^2 + 2^2)}{4!} \theta^4 + \frac{a(a^2 + 1)(a^2 + 3^2)}{5!} \theta^5 + \dots \infty$$

$$(840) \sum_{n=1}^{\infty} \frac{1}{(1 - n^2 m^2)^2}$$

$$(841) m\theta - \frac{m(m-1)(m-2)}{3!} \theta^3 \\ + \frac{m(m-1)(m-2)(m-3)(m-4)}{5!} \theta^5 + \dots \infty$$

$$(842) 1 - \frac{m(m-1)}{2!} \theta^2 + \frac{m(m-1)(m-2)(m-3)}{4!} \theta^4 + \dots \infty$$

$$(843) 1 + a\theta + \frac{a^2 - b^2}{2!} \theta^2 + \frac{a(a^2 - 3b^2)}{3!} \theta^3 \\ + \frac{a^4 - 6a^2b^2 + b^4}{4!} \theta^4 + \dots \\ + \frac{(a^2 + b^2)^{n/2}}{n!} \theta^n \cos \left(n \tan^{-1} \frac{b}{a} \right) + \dots \infty$$

$$(844) m\theta - \frac{m(m^2 - 1^2)}{3!} \theta^3 + \frac{m(m^2 - 1^2)(m^2 - 3^2)}{5!} \theta^5 - \dots \infty$$

$$= \tan^{-1} \theta \quad \text{where } |\theta(1 + \theta^2)|^2 < \frac{4}{27} \quad \text{A. 199}$$

$$= 2^{n/2} \cos \frac{n\pi}{4} \quad \text{where } n \text{ is positive} \quad \text{A. 311}$$

$$= -\log \cos \theta \quad \text{where } \theta^2 < \frac{\pi^2}{4} \quad \text{B. 245}$$

$$= e^{a \sin^{-1} \theta} \quad \text{L. 77}$$

$$= \frac{\pi^2}{4m^2} \operatorname{cosec}^2 \frac{\pi}{m} + \frac{\pi}{4m} \cot \frac{\pi}{m} - \frac{1}{2} \quad \begin{cases} \text{A. 217} \\ \text{A. 218} \end{cases}$$

$$= \sin (m \tan^{-1} \theta)(1 + \theta^2)^{m/2} \quad \text{L. 81}$$

$$= \cos (m \tan^{-1} \theta)(1 + \theta^2)^{m/2} \quad \text{L. 81}$$

$$= e^{a\theta} \cos b\theta \quad \text{L. 73}$$

$$= \sin (m \sin^{-1} \theta) \quad \text{L. 76}$$

Series No.

$$(845) \quad 1 - \frac{m^2 \theta^2}{2!} + \frac{m^2(m^2 - 2^2)}{4!} \theta^4 - \frac{m^2(m^2 - 2^2)(m^2 - 4^2)}{6!} \theta^6 + \dots \infty$$

$$(846) \quad \sum_{-\infty}^{+\infty} \left\{ \log h \left(1 + \frac{\theta}{n} \right) - \frac{\theta}{n} \right\}$$

$$(847) \quad \sum_{-\infty}^{+\infty} (-1)^n \left\{ \log h \left(1 + \frac{\theta}{n} \right) - \frac{\theta}{n} \right\}$$

$$(848) \quad \theta^2 + \left(1 - \frac{1}{3} + \frac{1}{5} \right) \frac{\theta^6}{3} + \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \right) \frac{\theta^{10}}{5} + \dots \infty$$

$$(849) \quad \mathcal{S}_2 \frac{\theta^3}{3} - \mathcal{S}_4 \frac{\theta^5}{5} + \mathcal{S}_6 \frac{\theta^7}{7} - \dots \infty$$

$$\mathcal{S}_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{2n}$$

$$(850) \quad \sum_{-\infty}^{+\infty} (-1)^n \left(\frac{1}{\theta - n} + \frac{1}{n} \right)$$

$$(851) \quad \sum_{-\infty}^{+\infty} \left\{ \frac{1}{n + x + (-1)^{n-1}y} - \frac{1}{n} \right\}$$

In Series Nos. (852) to (854), see No. (1134) for values of $A_n(x)$ and $B_n(x)$.

$$(852) \quad 2a \left(x - \frac{1}{2} \right) - \frac{(2a)^3}{2!} B_3(x) + \frac{(2a)^5}{4!} B_5(x) - \dots \infty$$

$$(853) \quad -(2a)^2 B_2(x) + \frac{(2a)^4}{3!} B_4(x) - \frac{(2a)^6}{5!} B_6(x) + \dots \infty$$

$$(854) \quad 1 - (2a)^2 A_2(x) + \frac{(2a)^4}{3!} A_4(x) - \frac{(2a)^6}{5!} A_6(x) + \dots \infty$$

$$(855) \quad \sum_1^{\infty} (-1)^{n-1} \frac{r! \theta^{2n-1}}{(2n-1)! [r - (2n-1)]!}$$

See No. (841).

$$= \cos (m \sin^{-1} \theta) \quad \text{L. 76}$$

$$= \log h \sin \pi \theta - \log h \pi \theta \quad \text{where } n \neq 0 \quad \text{A. 314}$$

$$= \log h \tan \frac{\pi \theta}{2} - \log h \frac{\pi \theta}{2} \quad \text{where } n \neq 0 \quad \text{A. 314}$$

$$= \frac{1}{2} \tan^{-1} \theta \log h \frac{1 + \theta}{1 - \theta} \quad \text{A. 191}$$

$$= \frac{1}{2} \tan^{-1} \theta \log h (1 + \theta^2) \quad \text{A. 191}$$

$$= \pi \operatorname{cosec} (\pi \theta) - \frac{1}{\theta} \quad \text{omit } n = 0$$

$$= \frac{\pi \cos \pi x}{\sin \pi x - \sin \pi y} - \frac{1}{x - y} \quad \text{omit } n = 0 \quad \text{A. 225}$$

$$= \frac{a \sin a(2x - 1)}{\sin a} \quad \text{AE. 25}$$

$$= \frac{a \cos a(2x - 1) - a \cos a}{\sin a} \quad \text{AE. 25}$$

$$= \frac{a \cos a(2x - 1)}{\sin a} \quad \text{AE. 25}$$

$$= (1 + \theta^2)^{r/2} \sin (r \tan^{-1} \theta) \quad \text{L. 81}$$

Series No.

$$(856) \sum_1^{\infty} (-1)^n \frac{r! \theta^{2n}}{(2n)!(r-2n)!}$$

See No. (842).

$$(857) \sum_{-\infty}^{+\infty} \frac{1}{n^4 + \theta^4}$$

$$(858) \sum_{-\infty}^{+\infty} \frac{1}{(n+x)^2 + y^2}$$

$$(859) \sum_1^{\infty} \frac{(-1)^n n}{(\epsilon^{nm} - \epsilon^{-nm})\{(nm)^4 + \frac{1}{4}\theta^4\}}$$

In Series Nos. (860) to (862), see No. (1134) for values of $A_n x$.

$$(860) 2aA_1(x) - \frac{(2a)^3}{2!} A_3(x) + \frac{(2a)^5}{4!} A_5(x) + \dots$$

$$(861) 2a \left\{ A_1(x) - 2A_1\left(\frac{x}{2}\right) \right\} - \frac{(2a)^3}{2!} \left\{ A_3(x) - 2^3 A_3\left(\frac{x}{2}\right) \right\} \\ + \frac{(2a)^5}{4!} \left\{ A_5(x) - 2^5 A_5\left(\frac{x}{2}\right) \right\} - \dots$$

$$(862) (2a)^2 \left\{ A_2(x) - 2^2 A_2\left(\frac{x}{2}\right) \right\} - \frac{(2a)^4}{3!} \left\{ A_4(x) - 2^4 A_4\left(\frac{x}{2}\right) \right\} \\ + \frac{(2a)^6}{5!} \left\{ A_6(x) - 2^6 A_6\left(\frac{x}{2}\right) \right\} - \dots$$

In series Nos. (863) to (873), see No. (330), etc. for values of p , q , r , and t .

$$(863) p_1 + q_2 a + p_3 a^2 + q_4 a^3 + \dots$$

$$(864) r_1 + t_2 a + r_3 a^2 + t_4 a^4 + \dots$$

Reference

$$= (1 + \theta^2)^{r/2} \cos (r \tan^{-1} \theta) - 1 \quad \text{L. 81}$$

$$= \frac{\pi \sinh \pi \theta \sqrt{2} + \sin \pi \theta \sqrt{2}}{\theta^3 \sqrt{2} \cosh \pi \theta \sqrt{2} - \cos \pi \theta \sqrt{2}} \quad \text{A. 313}$$

$$= \frac{\pi \sinh 2\pi y}{y \cosh 2\pi y - \cos 2\pi x} \quad \text{A. 314}$$

$$= \frac{1}{2\pi\theta^2 (\cosh \theta - \cos \theta)} - \frac{1}{2\pi\theta^4} \quad \text{Q. 135}$$

$$= \frac{a \sin (2x - 1)a}{\sin a} \quad \text{AE. 93}$$

$$= \frac{a \cos (2x - 1)a}{\cos a} \quad \text{AE. 93}$$

$$= \frac{a \sin (2x - 1)a}{\cos a} \quad \text{AE. 93}$$

$$= \frac{\pi}{4} \frac{1}{\sin \left(\frac{\pi}{4} - \frac{\pi a}{4} \right)} \quad \text{AE. 80}$$

$$= \frac{\pi}{6} \frac{1}{\sin \left(\frac{\pi}{6} - \frac{\pi a}{4} \right)} \quad \text{AE. 80}$$

Series No.

$$(865) \ p_1a + q_2a^2 + p_3a^3 + q_4a^4 + \dots$$

$$(866) \ r_1a + t_2a^2 + r_3a^3 + t_4a^4 + \dots$$

$$(867) \ p_1a + p_3a^3 + p_5a^5 + \dots$$

$$(868) \ r_1a + r_3a^3 + r_5a^5 + \dots$$

$$(869) \ t_2a^2 + t_4a^4 + t_6a^6 + \dots$$

$$(870) \ q_2a^2 + q_4a^4 + q_6a^6 + \dots$$

XVII. Hyperbolic Expansions

$$(871) \ 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots \infty = \sum_0^{\infty} \frac{\theta^{2n}}{(2n)!}$$

$$(872) \ \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \infty = \sum_0^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}$$

Reference

$$= \frac{\pi a}{2} \frac{\sin\left(\frac{\pi}{4} + \frac{\pi a}{4}\right)}{\cos \frac{\pi a}{2}} \quad \text{AE. 80}$$

$$= \frac{2\pi a}{3} \frac{\sin\left(\frac{\pi}{6} + \frac{\pi a}{6}\right)}{2 \cos \frac{\pi a}{3} - 1} \quad \text{AE. 80}$$

$$= \frac{\pi a}{2\sqrt{2}} \frac{\cos \frac{\pi a}{4}}{\cos \frac{\pi a}{2}} \quad \text{AE. 81}$$

$$= \frac{\pi a}{3} \frac{\cos \frac{\pi a}{6}}{2 \cos \frac{\pi a}{3} - 1} \quad \text{AE. 81}$$

$$= \frac{\pi a}{\sqrt{3}} \frac{\sin \frac{\pi a}{6}}{2 \cos \frac{\pi a}{3} - 1} \quad \text{AE. 81}$$

$$= \frac{\pi a}{2\sqrt{2}} \frac{\sin \frac{\pi a}{4}}{\cos \frac{\pi a}{2}} \quad \text{AE. 81}$$

$$= \cosh \theta \quad \text{where } \theta^2 < \infty \quad \text{E. 84}$$

$$= \sinh \theta \quad \text{where } \theta^2 < \infty \quad \text{E. 84}$$

Series No.

$$(873) \dagger \theta - \frac{\theta^3}{3} + \frac{2\theta^5}{15} - \frac{17\theta^7}{315} \dots \infty$$

$$= \sum_1^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} B_n \theta^{2n-1}$$

$$(874) 1 + \frac{m^2}{1^2} + \frac{m^2(m^2 + 1^2)}{1^2 \cdot 3^2} + \frac{m^2(m^2 + 1^2)(m^2 + 3^2)}{1^2 \cdot 3^2 \cdot 5^2} + \dots \infty$$

$$(875) \theta + \frac{\theta^3}{3} + \frac{\theta^5}{5} + \dots \infty$$

$$(876) \sum_{-\infty}^{+\infty} (-1)^{n-1} \frac{4n^2 - 1}{16n^4 + 4n^2 + 1}$$

$$(878) \ddagger 1 + \sum_1^{\infty} (-1)^{n-1} \frac{2^{2n} B_n \theta^{2n}}{(2n)!} = 1 + \frac{\theta^2}{3} - \frac{\theta^4}{45} + \frac{2}{945} \theta^6 - \dots \infty$$

$$(879) \dagger 1 + 2 \sum_1^{\infty} (-1)^n \frac{(2^{2n-1} - 1) B_n \theta^{2n}}{(2n)!}$$

$$= 1 - \frac{\theta^2}{6} + \frac{7\theta^4}{360} - \frac{31\theta^6}{15120} + \dots \infty$$

$$(880) \frac{1}{\theta} - 2\theta \left(\frac{1}{\pi^2 + \theta^2} - \frac{1}{4\pi^2 + \theta^2} + \frac{1}{9\pi^2 + \theta^2} - \dots \infty \right)$$

$$(881) 4\pi \left(\frac{1}{\pi^2 + 4\theta^2} - \frac{3}{9\pi^2 + 4\theta^2} + \frac{5}{25\pi^2 + 4\theta^2} - \dots \infty \right)$$

$$(882) \frac{1}{\theta} + 2\theta \left(\frac{1}{\pi^2 + \theta^2} + \frac{1}{4\pi^2 + \theta^2} + \frac{1}{9\pi^2 + \theta^2} + \dots \right)$$

$$(883) \sum_1^{\infty} \frac{(3^n - 3)\{1 - (-1)^n\}}{8n!} \theta^n$$

$$(884) \sum_1^{\infty} \frac{(3^n + 3)\{1 + (-1)^n\}}{8n!} \theta^n$$

† For values of B_n , see No. (1129).

‡ No. (877) has been omitted because it duplicates a previous series.—Ed.

$$= \tanh \theta \quad \text{where } \theta^2 < \frac{\pi}{4} \quad \text{H. 498}$$

$$= \cosh m\pi \quad \text{C. 421}$$

$$= \tanh^{-1} \theta \quad \text{where } |\theta| < 1 \quad \text{H. 475}$$

$$= \pi\sqrt{2} \cosh \frac{\pi\sqrt{3}}{4} \operatorname{sech} \frac{\pi\sqrt{3}}{2} \quad \text{A. 314}$$

$$= \theta \coth \theta \quad \text{C. 343}$$

$$= \theta \operatorname{cosech} \theta \quad \text{where } \theta^2 < \pi^2 \quad \text{C. 343}$$

$$= \operatorname{cosech} \theta \quad \text{Q. 136}$$

$$= \operatorname{sech} \theta \quad \text{Q. 136}$$

$$= \coth \theta \quad \text{Q. 136}$$

$$= \sinh^3 \theta \quad \text{Y. 80}$$

$$= \cosh^3 \theta \quad \text{Y. 80}$$

Series No.

$$(885) \frac{b}{a^2 + b^2} - \frac{b}{(2a)^2 + b^2} + \frac{b}{(3a)^2 + b^2} - \dots \infty$$

$$= \int_0^\infty \frac{\sin b\theta \, d\theta}{e^{a\theta} + 1}$$

$$(886) \frac{b}{a^2 + b^2} + \frac{b}{(2a)^2 + b^2} + \frac{b}{(3a)^2 + b^2} + \dots \infty$$

$$= \int_0^\infty \frac{\sin b\theta \, d\theta}{e^{a\theta} - 1}$$

$$(887) 1 - \frac{\theta^2}{3!} + \frac{14\theta^4}{6!} - \dots \infty$$

$$(888) \pi - 2 \left(\theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \dots \right)$$

$$(889) 1 + \frac{1^2 + 1^2}{2!} \sin^2 \theta + \frac{(1^2 + 1^2)(1^2 + 3^2)}{4!} \sin^4 \theta + \dots \infty$$

$$(890) \frac{1^2}{1} \sin \theta + \frac{(1^2 + 2^2)}{3!} \sin^3 \theta$$

$$+ \frac{(1^2 + 2^2)(1^2 + 4^2)}{5!} \sin^5 \theta + \dots \infty$$

$$(891) 1 + \frac{\theta^2}{2} - \frac{11\theta^4}{24} + \dots \infty$$

$$(892) \theta - \frac{\theta^3}{3} - \frac{\theta^5}{5} - \dots \infty$$

$$(893) 1 - \frac{2^2\theta^4}{4!} + \frac{2^4\theta^8}{8!} - \dots \infty$$

$$(894) \frac{2^2\theta^2}{2!} - \frac{2^4\theta^6}{6!} + \frac{2^6\theta^{10}}{10!} - \dots \infty$$

$$(895) \theta + \frac{2\theta^3}{3!} - \frac{2^2\theta^5}{5!} - \frac{2^3\theta^7}{7!} + \frac{2^4\theta^9}{9!} + \dots \infty$$

$$(896) \theta - \frac{2\theta^3}{3!} - \frac{2^2\theta^5}{5!} + \frac{2^3\theta^7}{7!} + \dots \infty$$

Reference

$$= \frac{1}{2} \left[\frac{1}{b} - \frac{\pi}{a \sinh \frac{\pi b}{a}} \right] \quad \text{A. 501}$$

$$= \frac{\pi}{a} \left[\frac{1}{e^{2\pi b/a} - 1} - \frac{a}{2\pi b} + \frac{1}{2} \right] \quad \text{A. 501}$$

$$= \frac{\theta}{\sinh \theta} \quad \text{H. 498}$$

$$= \cos^{-1} (\tanh \log \theta) \quad \text{L. 80}$$

$$= \frac{\cosh \theta}{\cos \theta} \quad \text{L. 81}$$

$$= \frac{\sinh \theta}{\cos \theta} \quad \text{L. 81}$$

$$= \cosh (\theta \cos \theta) \quad \text{L. 80}$$

$$= \sinh (\theta \cos \theta) \quad \text{L. 80}$$

$$= \cosh \theta \cdot \cos \theta \quad \text{H. 497}$$

$$= 2 \sinh \theta \cdot \sin \theta \quad \text{T. 127}$$

$$= \cosh \theta \cdot \sin \theta \quad \text{H. 497}$$

$$= \sinh \theta \cdot \cos \theta \quad \text{H. 497}$$

Series No.

$$(897) \quad 1 - \frac{\theta^2}{2} + \frac{5}{24}\theta^4 - \frac{61}{720}\theta^6 + \dots \infty = 1 + \sum_1^{\infty} (-1)^n \frac{E_n^*}{(2n)!} \theta^{2n}$$

For E_n^* see No. (1131).

$$(898) \quad \sum_0^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} \theta^{2n+1} \\ = \log \theta + \sum_0^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 2n} \theta^{-2n}$$

$$(899) \quad \log \theta - \sum_0^{\infty} \frac{(2n)!}{2^{2n}(n!)^2 2n} \theta^{-2n} \\ = \log \theta - \frac{1}{2} \cdot \frac{1}{2\theta^2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{4\theta^4} - \dots \infty$$

$$(900) \quad \sum_0^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} \theta^{-2n-1} \\ = \frac{1}{\theta} - \frac{1}{2} \cdot \frac{1}{3\theta^3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5\theta^5} - \dots \infty$$

$$(901) \quad \frac{\theta^2}{2} + \frac{2}{3} \cdot \frac{\theta^4}{4} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{\theta^6}{6} - \dots \infty$$

For values of the coefficients in Nos. (902) through (938), see No. (1142).

$$(902) \quad 2a \left(x - \frac{1}{2} \right) + \frac{(2a)^3}{2!} A_3(x) + \frac{(2a)^5}{4!} A_5(x) + \dots \infty$$

$$(903) \quad (2a)^2 B_2(x) + \frac{(2a)^4}{3!} B_4(x) + \frac{(2a)^6}{5!} B_6(x) + \dots \infty$$

$$(904) \quad 1 + (2a)^2 A_2(x) + \frac{(2a)^4}{3!} A_4(x) + \frac{(2a)^6}{5!} A_6(x) + \dots \infty$$

$$(905) \quad (2a)^2 B_2 \left(\frac{1}{4} \right) + \frac{(2a)^4}{3!} B_4 \left(\frac{1}{4} \right) + \dots \infty$$

Reference

$$= \operatorname{scch} \theta \quad \text{where } \theta^2 < \frac{\pi}{4}$$

T. 127

where $\theta^2 < 1$

$$= \sinh^{-1} \theta \quad \text{where } \theta^2 > 1$$

T. 128

$$= \cosh^{-1} \theta \quad \text{where } \theta^2 > 1$$

T. 128

$$= \sinh^{-1} \frac{1}{\theta} = \operatorname{cosech}^{-1} \theta \quad \text{where } \theta^2 > 1$$

T. 128

$$= \frac{(\sinh^{-1} \theta)^2}{2!}$$

Y. 90

$$= \frac{a \sinh a(2x - 1)}{\sinh a}$$

AE. 19

$$= \frac{a \cosh a(2x - 1) - a \cosh a}{\sinh a}$$

AE. 6

$$= \frac{a \cosh a(2x - 1)}{\sinh a}$$

AE. 19

$$= \frac{-2a \sinh \frac{1}{4}a \sinh \frac{3}{4}a}{\sinh a}$$

AE. 33

Series No.

$$(906) -\frac{1}{3}a + \frac{(2a)^3}{2!}B_3\left(\frac{1}{3}\right) + \frac{(2a)^5}{4!}B_5\left(\frac{1}{3}\right) + \dots \infty$$

$$(907) 1 + (2a)^2A_2\left(\frac{1}{4}\right) + \frac{(2a)^4}{3!}A_4\left(\frac{1}{4}\right) + \dots \infty$$

$$(908) a - \frac{(4a)^3}{2!}B_3\left(\frac{1}{4}\right) - \frac{(4a)^5}{4!}B_5\left(\frac{1}{4}\right) - \dots \infty$$

$$(909) \frac{1}{2}a - \frac{(3a)^3}{2!}B_3\left(\frac{1}{3}\right) - \frac{(3a)^5}{4!}B_5\left(\frac{1}{3}\right) - \dots \infty$$

$$(910) 2a - \frac{(6a)^3}{2!}B_3\left(\frac{1}{6}\right) - \frac{(6a)^5}{4!}B_5\left(\frac{1}{6}\right) - \dots \infty$$

$$(911) \frac{3a}{2} - \frac{(6a)^3}{2!}\left\{B_3\left(\frac{1}{6}\right) - \frac{1}{2^3}B_3\left(\frac{1}{3}\right)\right\} \\ - \frac{(6a)^5}{4!}\left\{B_5\left(\frac{1}{6}\right) - \frac{1}{2^5}B_5\left(\frac{1}{3}\right)\right\} - \dots \infty$$

$$(912) a - \frac{(6a)^3}{2!}B_3\left(\frac{1}{3}\right) - \frac{(6a)^5}{4!}B_5\left(\frac{1}{3}\right) - \dots \infty$$

$$(913) H_0 - \frac{H_1}{2!}(2a)^2 + \frac{H_2}{4!}(2a)^4 - \dots \infty$$

$$(914) 2 - \frac{a^2}{2!}\left\{8^3B_3\left(\frac{1}{8}\right) - 4^3B_3\left(\frac{1}{4}\right)\right\} \\ - \frac{a^4}{4!}\left\{8^5B_5\left(\frac{1}{8}\right) - 4^5B_5\left(\frac{1}{4}\right)\right\} - \dots \infty$$

$$(915) 1 + (2a)^2A_2\left(\frac{1}{4}\right) + \frac{(2a)^4}{3!}A_4\left(\frac{1}{4}\right) + \dots \infty$$

$$(916) \frac{1}{3} + (2-1)(3-1)B_1\frac{a^2}{2!} - (2^3-1)(3^3-1)B_2\frac{a^4}{4!} \\ + (2^5-1)(3^5-1)B_3\frac{a^6}{6!} + \dots$$

$$(917) \frac{1}{3} - 2(3-1)B_1\frac{a^2}{2!} + 2^3(3^3-1)B_2\frac{a^4}{4!} \\ - 2^5(3^5-1)B_3\frac{a^6}{6!} + \dots$$

Reference

$$= - \frac{a \sinh \frac{1}{3}a}{\sinh a} \quad \text{AE. 35}$$

$$= \frac{a}{2 \sinh \frac{1}{2}a} \quad \text{AE. 33}$$

$$= \frac{a}{\cosh a} \quad \text{AE. 32}$$

$$= \frac{3a \sinh \frac{1}{2}a}{2 \sinh \frac{3}{2}a} \quad \text{AE. 51}$$

$$= \frac{3a \sinh 2a}{\sinh 3a} \quad \text{AE. 43}$$

$$= \frac{3a \cosh \frac{1}{2}a}{2 \cosh \frac{3}{2}a} \quad \text{AE. 51}$$

$$= \frac{3a \sinh a}{\sinh 3a} \quad \text{AE. 35}$$

$$= \frac{3 \cosh a}{2 \cosh 3a} \quad \text{AE. 49}$$

$$= \frac{2 \cosh a}{\cosh 2a} \quad \text{AE. 60}$$

$$= \frac{a}{2} \operatorname{cosech} \frac{a}{2} \quad \text{AE. 33}$$

$$= \frac{a \cosh 2a}{\sinh 3a} \quad \text{AE. 47}$$

$$= \frac{a \cosh a}{\sinh 3a} \quad \text{AE. 47}$$

Series No.

$$(918) \quad 1 + (2a)^2 A_2 \left(\frac{1}{2}\right) + \frac{(2a)^4}{3!} A_4 \left(\frac{1}{2}\right) + \frac{(2a)^6}{5!} A_6 \left(\frac{1}{2}\right) + \dots$$

$$(919) \quad a - \frac{(4a)^3}{2!} B_3 \left(\frac{1}{4}\right) - \frac{(4a)^5}{4!} B_5 \left(\frac{1}{4}\right) - \dots$$

$$(920) \quad (2a)^2 B_2 \left(\frac{1}{2}\right) + \frac{(2a)^4}{3!} B_4 \left(\frac{1}{2}\right) + \frac{(2a)^6}{5!} B_6 \left(\frac{1}{2}\right) + \dots$$

$$(921) \quad 1 + B_1 \frac{a^2}{2!} - B_2 \frac{a^4}{4!} + B_3 \frac{a^6}{6!} - \dots$$

$$(922) \quad I_0 - \frac{I_1}{2!} a^2 + \frac{I_2}{4!} a^4 - \frac{I_3}{6!} a^6 + \dots$$

$$(923) \quad (8a)^2 B_2 \left(\frac{1}{4}\right) + \frac{(8a)^4}{3!} B_4 \left(\frac{1}{4}\right) + \dots$$

$$(924) \quad \frac{1}{2} a - \frac{(3a)^3}{2!} B_3 \left(\frac{1}{3}\right) - \frac{(3a)^5}{4!} B_5 \left(\frac{1}{3}\right) - \dots$$

$$(925) \quad \frac{3}{2} a - \frac{a^3}{2!} \left\{ 6^3 B_6 \left(\frac{1}{6}\right) - 3^3 B_3 \left(\frac{1}{3}\right) \right\} \\ - \frac{a^5}{4!} \left\{ 6^5 B_5 \left(\frac{1}{6}\right) - 3^5 B_5 \left(\frac{1}{3}\right) \right\} - \dots \infty.$$

$$(926) \quad (12a)^2 B_2 \left(\frac{1}{6}\right) + \frac{(12a)^4}{3!} B_4 \left(\frac{1}{6}\right) + \frac{(12a)^6}{5!} B_6 \left(\frac{1}{6}\right) + \dots \infty$$

$$(927) \quad - (12a) B_2 \left(\frac{1}{6}\right) - \frac{(12a)^3}{3!} B_4 \left(\frac{1}{6}\right) - \frac{(12a)^5}{5!} B_6 \left(\frac{1}{6}\right) - \dots \infty$$

$$(928) \quad \frac{1}{2} a - \frac{(2a)^3}{2!} B_3 \left(\frac{1}{4}\right) - \frac{(2a)^5}{4!} B_5 \left(\frac{1}{4}\right) - \dots \infty$$

$$(929) \quad \frac{1}{2} a - \frac{(3a)^3}{2!} B_3 \left(\frac{1}{3}\right) - \frac{(3a)^5}{4!} B_5 \left(\frac{1}{3}\right) - \dots \infty$$

$$(930) \quad \frac{3}{4} a - \frac{(2a)^3}{2!} B_3 \left(\frac{1}{8}\right) - \frac{(2a)^5}{4!} B_5 \left(\frac{1}{8}\right) - \dots \infty$$

$$(931) \quad (2a)^2 B_2 \left(\frac{1}{3}\right) + \frac{(2a)^4}{3!} B_4 \left(\frac{1}{3}\right) + \dots \infty$$

Reference

$= a \operatorname{cosech} a$	AE. 30
$= a \operatorname{sech} a$	AE. 32
$= -a \tanh \frac{1}{2} a$	AE. 30
$= \frac{1}{2} a \coth \frac{1}{2} a$	AE. 47
$= \frac{3(\sinh 2a - \sinh a)}{2 \sinh 3a}$	AE. 45
$= -\frac{8a \sinh a \sinh 3a}{\sinh 4a}$	AE. 33
$= \frac{3a}{2(1 + 2 \cosh a)}$	AE. 37
$= \frac{3a}{2(2 \cosh a - 1)}$	AE. 48
$= \frac{-12a \sinh a \sinh 5a}{\sinh 6a}$	AE. 46
$= \frac{\sinh 5a}{2(\cosh a + \cosh 3a + \cosh 5a)}$	AE. 46
$= \frac{a \sinh \frac{1}{2}a}{\sinh a}$	AE. 60
$= \frac{3a \sinh \frac{1}{2}a}{2 \sinh \frac{3}{2}a}$	AE. 48
$= \frac{a \sinh \frac{3}{4}a}{\sinh a}$	AE. 60
$= -\frac{2a \sinh \frac{1}{3}a \sinh \frac{2}{3}a}{\sinh a}$	AE. 40

Series No.

$$(932) -\frac{2}{3}a + \frac{(2a)^3}{2!}B_3\left(\frac{1}{6}\right) + \frac{(2a)^5}{4!}B_5\left(\frac{1}{6}\right) + \dots \infty$$

$$(933) -\frac{1}{3}a + \frac{(2a)^3}{2!}B_3\left(\frac{1}{3}\right) + \frac{(2a)^5}{4!}B_5\left(\frac{1}{3}\right) + \dots \infty$$

$$(934)^\dagger t_2a^2 - t_4a^4 + t_6a^6 - \dots \infty$$

$$(935) (2a)^2B_2\left(\frac{1}{4}\right) + \frac{(2a)^4}{3!}B_4\left(\frac{1}{4}\right) + \dots \infty$$

$$(936) 2a\left(x - \frac{1}{2}\right) + \frac{(2a)^3}{2!}A_3(x) + \frac{(2a)^5}{4!}A_5(x) + \dots \infty$$

$$(937) 1 + (2a)^2A_2(x) + \frac{(2a)^4}{3!}A_4(x) + \frac{(2a)^6}{5!}A_6(x) + \dots \infty$$

$$(938)^\dagger p_1a - p_3a^3 + p_5a^5 - \dots \infty$$

$$(939) \log_h \frac{2}{\theta} - \sum_0^{\infty} \frac{(2n)!}{2^{2n}(n!)^2 2n} \theta^{2n} \\ = \log_h \frac{2}{\theta} - \frac{1}{2} \cdot \frac{\theta^2}{2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\theta^4}{4} - \dots \infty$$

$$(940) \log_h \frac{2}{\theta} + \sum_0^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 2n} \theta^{2n} \\ = \log_h \frac{2}{\theta} + \frac{1}{2} \cdot \frac{\theta^2}{2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\theta^4}{4} \dots \infty$$

$$(941) \sum_0^{\infty} \frac{\theta^{-2n-1}}{2n+1} = \frac{1}{\theta} + \frac{1}{3\theta^3} + \frac{1}{5\theta^5} + \dots \infty$$

$$(942) \sum_0^{\infty} \epsilon^{-\theta(2n+1)}$$

† For values of p and t , see No. (330).

Reference

$$= - \frac{a \sinh \frac{3}{4}a}{\sinh a} \quad \text{AE. 43}$$

$$= - \frac{a \sinh \frac{1}{4}a}{\sinh a} \quad \text{AE. 35}$$

$$= \frac{\pi a}{\sqrt{3}} \frac{\sinh \frac{\pi a}{6}}{2 \cosh \frac{\pi a}{3} - 1} \quad \text{AE. 75}$$

$$= - \frac{2a \sinh \frac{1}{4}a \sinh \frac{3}{4}a}{\sinh a} \quad \text{AE. 33}$$

$$= \frac{a \sinh a(2x - 1)}{\sinh a} \quad \text{AE. 19}$$

$$= \frac{a \cosh a(2x - 1)}{\sinh a} \quad \text{AE. 19}$$

$$= \frac{\pi a}{2\sqrt{2}} \cdot \frac{\cosh \frac{\pi a}{4}}{\cosh \frac{\pi a}{2}} \quad \text{AE. 60}$$

$$= \cosh^{-1} \frac{1}{\theta} = \operatorname{sech}^{-1} \theta \quad \text{where } \theta^2 < 1 \quad \text{T. 128}$$

$$= \sinh^{-1} \frac{1}{\theta} = \operatorname{cosech}^{-1} \theta \quad \text{where } \theta^2 < 1 \quad \text{T. 128}$$

$$= \tanh^{-1} \frac{1}{\theta} = \operatorname{coth}^{-1} \theta \quad \text{where } \theta^2 > 1 \quad \text{T. 128}$$

$$= \frac{1}{2 \sinh \theta} \quad \text{T. 129}$$

Series No.

$$(943) \sum_0^{\infty} (-1)^n \epsilon^{-\theta(2n+1)}$$

$$(944) \sum_1^{\infty} (-1)^n \epsilon^{-2n\theta}$$

$$(945) \sum_0^{\infty} \frac{1}{2n+1} \epsilon^{-\theta(2n+1)}$$

$$(946) \sum_1^{\infty} \left\{ \frac{1}{(2n\pi - a)^2 + \theta^2} + \frac{1}{(2n\pi + a)^2 + \theta^2} \right\}$$

$$(947) 1 + \frac{2}{1+1^2} + \frac{2}{1+2^2} + \frac{2}{1+3^2} + \dots \infty$$

$$(948) 1 + \frac{2}{1+2^2} + \frac{2}{1+4^2} + \frac{2}{1+6^2} + \dots \infty$$

$$(949) \frac{1}{2^2+1^2} + \frac{1}{2^2+3^2} + \frac{1}{2^2+5^2} + \dots \infty$$

$$(950) \frac{1}{1+1^2} + \frac{1}{1+3^2} + \frac{1}{1+5^2} + \dots \infty$$

$$(951) \frac{1}{2} + \theta^2 \sum_1^{\infty} \frac{1}{\theta^2 + n^2 \pi^2}$$

$$(952) \sum_{-\infty}^{\infty} \frac{1}{(n+\theta)^2 + y^2}$$

$$(953) \frac{\theta^2}{6} - \frac{\theta^4}{180} + \dots \infty$$

$$= B_1^* \frac{2^2 \cdot \theta^2}{2 \cdot 2!} - B_3^* \frac{2^4 \cdot \theta^4}{4 \cdot 4!} + B_5^* \frac{2^6 \cdot \theta^6}{6 \cdot 6!} - \dots \infty$$

$$(954) 1 + \frac{n\theta^2}{2!} + \frac{n(3n-2)}{4!} \theta^4 + \dots \infty$$

$$(956)^\dagger \theta - \frac{2}{3} \theta^3 + \frac{2 \cdot 4}{3 \cdot 5} \theta^5 - \dots$$

† No. (955) has been omitted because it duplicates a previous series.—Ed.

Reference

$= \frac{1}{2 \cosh \theta}$		T. 129
$= \frac{1}{2} (\tanh \theta - 1)$		T. 129
$= -\frac{1}{2} \log \tanh \frac{\theta}{2}$		T. 129
$= \frac{1}{2\theta} \frac{\sinh \theta}{\cosh \theta - \cos a} - \frac{1}{a^2 + \theta^2}$		A. 314
$= \pi \coth \pi$	$1 - i\psi(1+i) + i\psi(1-i)$	Y. 55
$= \frac{\pi}{2} \coth \frac{\pi}{2}$		Y. 55
$= \frac{\pi}{8} \tanh \pi$		Y. 55
$= \frac{\pi}{4} \tanh \frac{\pi}{2}$		Y. 55
$= \frac{\theta}{2} \coth \theta$		Y. 55
$= \frac{\pi}{y} \frac{\sinh 2\pi y}{\cosh 2\pi y - \cos 2\pi \theta}$		A. 314
$= \log \frac{\sinh \theta}{\theta}$		Y. 109
$= \cosh^n \theta$		Y. 80
$= \frac{\sinh^{-1} \theta}{\sqrt{1 + \theta^2}}$		Y. 90

Series No.

XVIII. Taylor's and Maclaurin's Theorem

$$(957) \phi(a) + x\phi'(a) + \frac{x^2}{2!}\phi''(a) + \dots \frac{x^n}{n!}\phi^n(a) + \dots \infty$$

$$(958) \phi(0) + x\phi'(0) + \frac{x^2}{2!}\phi''(0) + \dots \frac{x^n}{n!}\phi^n(0) + \dots \infty$$

XIX. Bessel Functions

$$(959) \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r} r! (n+r)!}$$

$$(960) 1 - \frac{x^2}{2 \cdot 2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots \infty$$

$$(961) \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \dots \infty$$

$$(962) \frac{x^n}{2^n \cdot n!} \left\{ 1 - \frac{x^2}{2^2 \cdot 1 \cdot (n+1)} + \frac{x^4}{2^4 \cdot 1 \cdot 2(n+1)(n+2)} - \dots \infty \right\}$$

$$(963) \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r} r! \Gamma(n+r+1)}$$

$$(964) \sqrt{\frac{2x}{\pi}} \left\{ 1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4 \cdot 5} - \dots \infty \right\}$$

$$(965) \left(\frac{2}{\pi x} \right)^{1/2} \cos x$$

$$(966) \frac{x^n e^{n\sqrt{1-x^2}}}{\sqrt{2\pi n} (1-x^2)^{1/4} \{1 + \sqrt{1-x^2}\}^n}$$

XX. Elliptic Functions

$$(967) \frac{\pi}{2} \left[1 + \left(\frac{1}{2} \right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \dots \infty \right]$$

$$(968) \frac{\pi}{2} \left[1 - \left(\frac{1}{2} \right)^2 \frac{k^2}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \frac{k^4}{3} - \dots \infty \right]$$

Reference

$$= \phi(a + x) \quad \text{H. 481}$$

$$= \phi(x) \quad \text{H. 480}$$

$$= J_n(x) \quad \text{Q. 355}$$

$$= J_0(x) \quad \text{Q. 355}$$

$$= J_1(x) \quad \text{Q. 355}$$

$$= J_n(x) \quad \text{Q. 355}$$

$$= J_n(x) \quad \text{when } n \text{ is any general value} \quad \text{Q. 359}$$

$$= \left(\frac{2}{\pi x} \right)^{1/2} \sin x = J_{1/2}(x) \quad \text{Q. 364}$$

$$= J_{-1/2}(x) \quad \text{Q. 364}$$

$$\approx J_n(nx) \quad \text{when } n \text{ is large and } 0 < x < 1 \quad \text{Q. 369}$$

$$= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \left. \vphantom{\int_0^{\pi/2}} \right\} \quad \text{where } 0 < k < 1 \quad \text{A. 190}$$

$$= \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \quad \text{where } 0 < k < 1 \quad \text{A. 190}$$

Series No.

XXI. Various Integrals

$$(969) \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

$$(970) x \log 2 + \frac{1}{2} \left\{ \frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \dots \infty \right\}$$

$$(971) x \log 2 - \frac{1}{2} \left\{ \frac{\sin 2x}{1^2} - \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \dots \infty \right\}$$

$$(972) \frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \dots \infty$$

$$(973) x - \frac{1}{3}x^3 + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots \infty = \sum_0^{\infty} \frac{(-1)^k}{k!(2k+1)} x^{2k+1}$$

$$(974) x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \dots \infty \\ = \sum_0^{\infty} \frac{(-1)^k}{(2k)!(4k+1)} x^{4k+1}$$

$$(975) \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots \infty$$

$$(976) \frac{1}{e} (1 - 2! + 3! - \dots \infty) = \frac{0.4036526}{e}$$

$$(977) \log x + \frac{ax}{1} + \frac{a^2x^2}{1 \cdot 2^2} + \frac{a^3x^3}{1 \cdot 2 \cdot 3^2} + \dots \infty$$

$$(978) e^{-ab} \left[\log(b+x) + \frac{a(b+x)}{1 \cdot 1} + \frac{a^2(b+x)^2}{2 \cdot 2!} + \dots \infty \right]$$

$$(979) \log x + x \log a + \frac{(x \log a)^2}{2 \cdot 2!} + \dots \infty$$

$$(980) \frac{a^x x^n}{\log a} - \frac{na^x x^{n-1}}{(\log a)^2} + \frac{n(n-1)a^x x^{n-2}}{(\log a)^3} + \dots \\ \pm \frac{n(n-1)(n-2) \dots 2 \cdot 1 \cdot a^x}{(\log a)^{n+1}}$$

$$= \int_0^{\pi/2} \cos^n \theta \, d\theta = \int_0^{\pi/2} \sin^n \theta \, d\theta \quad \text{where } n \text{ is even} \quad \text{J. 48}$$

$$= - \int_0^x \log \sin x \, dx \quad \text{where } 0 < x < \pi \quad \text{X. 142}$$

$$= - \int_0^x \log \cos x \, dx \quad \text{where } -\frac{\pi}{2} < x < \frac{\pi}{2} \quad \text{X. 142}$$

$$= - \int_0^x \log \tan x \, dx \quad \text{where } 0 < x < \frac{\pi}{2} \quad \text{X. 142}$$

$$= \int_0^x e^{-x^2} \, dx \quad \text{where } x^2 < \infty$$

$$= \int_0^x \cos(x^2) \, dx \quad \text{where } x^2 < \infty$$

T. 133
See also
A. 336
T. 134

$$= \int_0^1 \frac{x^{a-1} dx}{1+x^b} \quad \text{T. 134}$$

$$= \int_1^\infty \frac{e^{-t}}{t} \, dt = \int_0^1 \frac{e^{-1/y}}{y} \, dy \quad \text{A. 336}$$

$$= \int \frac{e^{ax} \, dx}{x}$$

$$= \int \frac{e^{ax} \, dx}{b+x}$$

$$= \int \frac{a^x \, dx}{x}$$

$$= \int x^n a^x \, dx$$

Series No.

$$(981) \quad x + \frac{x^3}{3 \cdot 3!} + \frac{7x^5}{3 \cdot 5 \cdot 5!} + \frac{31x^7}{3 \cdot 7 \cdot 7!} + \frac{127x^9}{3 \cdot 5 \cdot 9!} + \dots \infty$$

$$(982) \quad 1 + \frac{x}{2^p} + \frac{x^2}{3^p} + \frac{x^3}{4^p} + \dots \infty$$

$$(983) \quad \frac{1}{1+1^2} + \frac{x}{1+2^2} + \frac{x^2}{1+3^2} + \dots \infty$$

$$(985) \dagger \log \theta + \theta + \frac{\theta^2}{2 \cdot 2!} + \frac{\theta^3}{3 \cdot 3!} + \dots \infty$$

$$(986) \quad \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3}$$

$$(987) \quad \theta - \frac{\theta^3}{3 \cdot 3!} + \frac{\theta^5}{5 \cdot 5!} - \frac{\theta^7}{7 \cdot 7!} + \dots \infty$$

$$(988) \quad \log \theta - \frac{\theta^2}{2 \cdot 2!} + \frac{\theta^4}{4 \cdot 4!} - \dots \infty$$

$$(989) \quad \frac{\pi}{2} \log 2 = 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\pi}{4n}$$

$$(990) \quad 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$(991) \quad \frac{\pi^2}{4} - 2 \left[(1+a)\epsilon^{-a} + \frac{1}{3^2} (1+3a)\epsilon^{-3a} \right. \\ \left. + \frac{1}{5^2} (1+5a)\epsilon^{-5a} + \dots \infty \right]$$

$$(992) \quad -C - \log |x| + x - \frac{1}{2} \frac{x^2}{2!} + \frac{1}{3} \frac{x^3}{3!} - \frac{1}{4} \frac{x^4}{4!} + \dots \infty$$

C = Euler's constant; see No. (1132).

† No. (984) has been omitted because it duplicates a previous series.—Ed.

Reference

$$= \int \frac{x \, dx}{\sin x}$$

AA. 204

$$= \frac{1}{\Gamma(p)} \int_0^1 \frac{\left[\log \frac{1}{\xi} \right]^{p-1}}{1 - x\xi} d\xi \quad \text{where } p > 0$$

A. 294
(1907)

$$= \int_0^1 \frac{\sin \left[\log \frac{1}{\xi} \right]}{1 - x\xi} d\xi$$

A. 294
(1907)

$$= \int \frac{e^\theta}{\theta} d\theta$$

K. 305

$$= \int_0^{\pi/2} \cos^n \theta \, d\theta = \int_0^{\pi/2} \sin^n \theta \, d\theta \quad \text{where } n \text{ is odd}$$

1Z. 226

$$= \int \frac{\sin \theta}{\theta} d\theta$$

K. 305

$$= \int \frac{\cos \theta}{\theta} d\theta$$

K. 305

$$= \int_0^{\pi/2} \theta \cot \theta \, d\theta$$

2Z. 187

$$= \int_0^{\pi/2} \frac{\theta \, d\theta}{\sin \theta} \quad \text{See also No. (308).}$$

2Z. 246

$$= \int_0^{\pi/2} \tan^{-1} (\sinh a \sin \theta) d\theta = \int_0^a \frac{\theta \, d\theta}{\sinh \theta} \quad \text{where } a \text{ is positive}$$

A. 517

$$= \int_x^\infty \frac{e^{-t}}{t} dt$$

A. 334

Series No.

$$(993) \frac{1}{x} - \frac{1}{1!} \frac{1}{x+1} + \frac{1}{2!} \frac{1}{x+2} - \frac{1}{3!} \frac{1}{x+3} + \dots \infty$$

$$(994) \sum_0^{\infty} \frac{1 \cdot 3 \dots 2n-1}{2 \cdot 4 \dots 2n} \cdot \frac{1}{(2n+1)^2}$$

$$(995) \sum_0^{\infty} (-1)^n \frac{1}{(2n+1)^2} \quad \text{See also No. (308).}$$

$$(996) \frac{x^2}{1 \cdot 2} - \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} - \dots \infty$$

$$(997) - \sum_1^{\infty} \frac{x^n}{n^2}$$

$$(998) \sum_1^{\infty} (-1)^{n-1} \frac{x^n}{n^2}$$

$$(999) 2 \sum_1^{\infty} \frac{x^{2n-1}}{(2n-1)^2}$$

$$(1000) \frac{1}{a} - \frac{1}{a+1} + \frac{1}{a+2} - \dots \infty$$

$$(1001) \frac{1}{(n-1)!} \cdot \frac{\pi}{2^n} \left[n^{n-1} - n(n-2)^{n-1} \right. \\ \left. + \frac{n(n-1)}{2!} (n-4)^{n-1} + \dots \right]$$

The number of terms in the brackets is $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$.

$$(1002) 1 - \frac{x}{2^2} + \frac{x^2}{3^3} - \frac{x^3}{4^4} + \frac{x^4}{5^5} - \dots \infty$$

$$(1003) e^{\theta} \sum_{r=0}^{r=n} (-1)^r \frac{n!}{(n-r)!} \theta^{n-r} 2^{-(r+1)/2} \sin \left\{ \theta - \frac{(r+1)\pi}{4} \right\}$$

$$(1004) \frac{1}{e} \left\{ 1 + \frac{1}{3 \cdot 1!} + \frac{1}{5 \cdot 2!} + \frac{1}{7 \cdot 3!} + \dots \infty \right\}$$

Reference

$$= \int_0^1 e^{-t} t^{x-1} dt \quad \text{where } x > 1$$

Q. 260

$$= \int_0^1 \frac{\sin^{-1} x}{x} dx \quad \text{where } |x| < 1$$

AB. 165

$$= \int_0^1 \frac{\tan^{-1} x}{x} dx \quad \text{where } |x| < 1$$

AB. 165

$$= \int_0^x \log(1+x) dx \quad \text{where } |x| < 1$$

AB. 166

$$= \int_0^x \log(1-x) \frac{dx}{x} \quad \text{where } 0 < x < 1$$

AB. 166

$$= \int_0^x \log(1+x) \frac{dx}{x} \quad \text{where } 0 < x < 1$$

AB. 166

$$= \int_0^x \log \frac{1+x}{1-x} \cdot \frac{dx}{x} \quad \text{where } 0 < x < 1$$

AB. 166

$$= \int_0^1 \frac{x^{a-1}}{1+x} dx \quad \text{where } a > 0$$

A. 189

$$= \int_0^\infty \left(\frac{\sin \theta}{\theta} \right)^n d\theta$$

A. 518

$$= \int_0^1 v^{\nu x} dv$$

1Z. 135

$$= \int \theta^n e^\theta \sin \theta d\theta$$

1Z. 135

$$= \int_0^{\pi/2} e^{-\sin 2\theta} \cos \theta d\theta$$

1Z. 638

Series No.

$$(1005) \quad 1^2 + \left(\frac{1}{1!}\right)^2 + \left(\frac{1}{2!}\right)^2 + \left(\frac{1}{3!}\right)^2 + \dots \infty$$

$$(1006) \quad 8 \left[-\frac{a}{1 \cdot 1 \cdot 3} + \frac{a^3}{1 \cdot 3 \cdot 5} + \frac{a^5}{3 \cdot 5 \cdot 7} + \frac{a^7}{5 \cdot 7 \cdot 9} + \dots \infty \right]$$

$$(1007) \quad \frac{\pi}{2^n} \frac{e^{-ar}}{(n-1)!} \left[r^{n-1} a^{-n} + \frac{n(n-1)}{2} r^{n-2} a^{-(n+1)} \right. \\ \left. + \frac{(n+1)n(n-1)(n-2)}{2 \cdot 4} r^{n-3} a^{-(n+2)} + \dots n \text{ terms} \right]$$

XXII. Beta and Gamma Functions. See also (1101).

$$(1008) \quad \int_0^1 x^{l-1} (1-x)^{m-1} dx$$

$$\int_0^\infty e^{-x} x^{n-1} dx$$

$$B(l, m)$$

$$\Gamma(n+1)$$

$$\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{3}{n}\right) \dots \Gamma\left(\frac{n-1}{n}\right)$$

$$\Gamma(x) \Gamma\left(x + \frac{1}{n}\right) \Gamma\left(x + \frac{2}{n}\right) \dots \Gamma\left(x + \frac{n-1}{n}\right)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$(1009) \quad \sum_0^\infty \frac{2n!}{2^{2n}(n!)^2} \frac{1}{x+n}$$

$$(1010) \quad \sum_0^\infty (-1)^n \frac{a(a-1)(a-2)\dots(a-n)}{n!} \frac{1}{x+n}$$

$$(1011) \quad e^{-x} x^{x-1/2} (2\pi)^{1/2} \left\{ 1 + \frac{1}{12x} + \frac{1}{288x^2} \right. \\ \left. - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + 0 \frac{1}{x^5} \right\}$$

Reference

$$= \frac{2}{\pi} \int_0^{\pi} e^{2 \cos \theta} \cos^2 (\sin \theta) d\theta - 1 \quad \text{2Z. 296}$$

$$= \int_0^{\pi} \sin 2\theta \log (1 - 2a \cos \theta + a^2) d\theta \quad \text{where } a^2 < 1 \quad \text{2Z. 308}$$

$$= \int_0^{\infty} \frac{\cos rx}{(a^2 + x^2)^n} dx \quad \text{2Z. 227}$$

$$= B(l, m) \quad (\text{Beta function})$$

$$= \Gamma(n) \quad (\text{Gamma function})$$

$$= \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}$$

$$= n\Gamma(n) = n! \quad \text{where } n \text{ is a positive integer}$$

$$= (2\pi)^{(n-1)/2} n^{-1/2} \quad \text{2Z. 62}$$

$$= \Gamma(nx)(2\pi)^{(n-1)/2} n^{1/2-nx} \quad \text{2Z. 94}$$

$$\Gamma(0) = \infty \quad \text{2Z. 59}$$

$$= \frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x + \frac{1}{2})} \quad \text{Q. 259}$$

$$= \frac{\Gamma(x)\Gamma(a+1)}{\Gamma(x+a)} \quad \text{where } a \text{ is positive} \quad \text{Q. 260}$$

$$= \Gamma(x) \quad \text{Q. 253}$$

Series No.

$$(1012) \frac{1}{x} + \frac{1^2}{4} \cdot \frac{1}{x(x+1)} + \frac{1^2 \cdot 3^2}{4 \cdot 8} \frac{1}{x(x+1)(x+2)} + \dots \infty$$

$$(1013) \frac{1}{x} - \frac{a-1}{x+1} + \frac{(a-1)(a-2)}{2!(x+2)} - \frac{(a-1)(a-2)(a-3)}{3!(x+3)} + \dots \infty$$

$$(1014) 1 + \frac{a}{b}(x) + \frac{a(a+1)}{b(b+1)}x^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \dots \infty$$

(1015) If in No. (1014), $b = 1$

$$1 + ax + \frac{a(a+1)}{2!}x^2 + \dots \infty$$

XXIII. Infinite Products

$$(1016) \theta \left(1 - \frac{\theta^2}{\pi^2}\right) \left(1 - \frac{\theta^2}{2^2\pi^2}\right) \left(1 - \frac{\theta^2}{3^2\pi^2}\right) \dots \infty$$

$$(1017) \left(1 - \frac{4\theta^2}{\pi^2}\right) \left(1 - \frac{4\theta^2}{3^2\pi^2}\right) \left(1 - \frac{4\theta^2}{5^2\pi^2}\right) \dots \infty$$

$$(1018) 2^{n-1} \left\{ \cosh \phi - \cos \theta \right\} \left\{ \cosh \phi - \cos \left(\theta + \frac{2\pi}{n} \right) \right\} \dots \left\{ \cosh \phi - \cos \left(\theta + \frac{2n-2}{n} \pi \right) \right\} \\ = 2^{n-1} \prod_{r=0}^{r=n-1} \left\{ \cosh \phi - \cos \left(\theta + \frac{2r\pi}{n} \right) \right\}$$

$$(1019) \prod_{r=0}^{r=\frac{n-2}{2}} \left(x^2 - 2x \cos \frac{2r+1}{n} \pi + 1 \right)$$

$$(1020) (x+1) \prod_{r=0}^{r=\frac{n-3}{2}} \left\{ x^2 - 2x \cos \frac{2r+1}{n} \pi + 1 \right\}$$

$$(1021) 2^{(n-1)/2} \sin \frac{2\pi}{2n} \sin \frac{4\pi}{2n} \dots \sin \frac{n-2}{2n} \pi$$

Reference

$$= \left[\frac{\Gamma(x)}{\Gamma(x + \frac{1}{2})} \right]^2 \quad \text{where } x \text{ is positive}$$

A. 524

$$= \frac{\Gamma(x)\Gamma(a)}{\Gamma(x+a)} \quad \text{where } x \text{ and } a \text{ are positive}$$

A. 524

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \frac{t^{a-1}(1-t)^{b-a-1}}{1-xt} dt \quad \text{where } b > a > 0$$

A. 294
(1907)

$$= (1-x)^{-a}$$

$$= \sin \theta$$

A. 213

$$= \cos \theta$$

A. 214

$$= \cosh n\phi - \cos n\theta$$

E. 143

$$= x^n + 1 \quad \text{where } n \text{ is even}$$

E. 143

$$= x^n + 1 \quad \text{where } n \text{ is odd}$$

E. 143

$$= \sqrt[n]{n} \quad \text{where } n \text{ is even}$$

E. 144

Series No.

$$(1022) \quad 2^{(n-1)/2} \cos \frac{2\pi}{2n} \cos \frac{4\pi}{2n} \dots \cos \frac{n-1}{2n} \pi$$

$$(1023) \quad \cos \theta \cos \left(\theta + \frac{2\pi}{n} \right) \dots \cos \left\{ \theta + (n-1) \frac{2\pi}{n} \right\}$$

$$(1024) \quad \sin \theta \sin \left(\theta + \frac{2\pi}{n} \right) \dots \sin \left\{ \theta + (n-1) \frac{2\pi}{n} \right\}$$

$$(1025) \quad \tan \frac{\pi}{n} \tan \frac{2\pi}{n} \dots \tan \frac{\frac{1}{2}(n-1)\pi}{n}$$

$$(1026) \quad 2^{n-1} \left(\cos \theta - \cos \frac{\pi}{2n} \right) \left(\cos \theta - \cos \frac{3\pi}{2n} \right) \dots \\ \left(\cos \theta - \cos \frac{2n-1}{2n} \pi \right)$$

$$(1027) \quad 2^{(n-1)/2} \sin \frac{\pi}{2n} \sin \frac{3\pi}{2n} \dots \sin \frac{n-2}{2n} \pi \\ = 2^{(n-1)/2} \cos \frac{2\pi}{2n} \cos \frac{4\pi}{2n} \dots \cos \frac{n-1}{2n} \pi$$

$$(1028) \quad (1-x) \left(1 + \frac{1}{2}x \right) \left(1 - \frac{1}{3}x \right) \left(1 + \frac{1}{4}x \right) \dots \infty$$

$$(1029) \quad (1-x) \left(1 + \frac{1}{3}x \right) \left(1 - \frac{1}{5}x \right) \left(1 + \frac{1}{7}x \right) \dots \infty$$

$$(1030) \quad \pi x \left(1 + \frac{x}{1} \right) \left(1 + \frac{x}{2} \right) \left(1 + \frac{x}{3} \right) \left(1 - \frac{x}{1} \right) \\ \times \left(1 + \frac{x}{4} \right) \left(1 + \frac{x}{5} \right) \left(1 + \frac{x}{6} \right) \left(1 - \frac{x}{2} \right) \dots$$

$$(1031) \quad \prod_{-\infty}^{+\infty} \left\{ e^{\theta/n\pi} \left(1 - \frac{\theta}{n\pi} \right) \right\}$$

Reference

$$= 1 \quad \text{where } n \text{ is odd}$$

E. 144

$$= \frac{1}{2^{n-1}} \cos n\theta \quad \text{where } n \text{ is odd}$$

$$= \frac{1}{2^{n-1}} [(-1)^{n/2} - \cos n\theta] \quad \text{where } n \text{ is even}$$

E. 73

$$= (-1)^{(n-1)/2} \frac{1}{2^{n-1}} \sin n\theta \quad \text{where } n \text{ is odd}$$

$$= (-1)^{n/2} \frac{1}{2^{n-1}} (1 - \cos n\theta) \quad \text{where } n \text{ is even}$$

E. 73

$$= \sqrt{n} \quad \text{where } n \text{ is odd}$$

E. 145

$$= \cos n\theta$$

E. 145

$$= 1 \quad \text{where } n \text{ is odd}$$

E. 145

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(1 + \frac{1}{2}x)\Gamma(\frac{1}{2} - \frac{1}{2}x)}$$

A. 115

$$= \cos \frac{\pi x}{4} - \sin \frac{\pi x}{4}$$

A. 224

$$= e^{x \log h^3} \sin \pi x$$

AG.40

$$= \frac{\sin \theta}{\theta} \quad \text{omitting } n = 0$$

A. 215

Series No.

$$(1032) \prod_{n=1}^{n=\infty} \left\{ \left(1 - \frac{\theta}{n\pi} \right) e^{\theta/n\pi} \left(1 + \frac{\theta}{n\pi} \right) e^{-\theta/n\pi} \right\}$$

$$(1033) \prod_{-\infty}^{+\infty} \left\{ e^{2\theta/(2n+1)\pi} \left(1 - \frac{2\theta}{(2n+1)\pi} \right) \right\}$$

$$(1034) \prod_2^{\infty} \frac{n^3 - 1}{n^3 + 1}$$

$$(1035) \prod_1^{\infty} \frac{\left(1 + \frac{1}{r} \right)^x}{1 + \frac{x}{r}}$$

$$(1036) x! \left(x - \frac{1}{n} \right)! \left(x - \frac{2}{n} \right)! \dots \left(x - \frac{n-1}{n} \right)!$$

$$(1037) \left\{ \left(1 + \frac{k}{\theta} \right)^2 \right\} \left\{ 1 + \left(\frac{k}{2\pi - \theta} \right)^2 \right\} \\ \left\{ 1 + \left(\frac{k}{2\pi + \theta} \right)^2 \right\} \left\{ 1 + \left(\frac{k}{4\pi - \theta} \right)^2 \right\} \left\{ 1 + \left(\frac{k}{4\pi + \theta} \right)^2 \right\} \dots \infty$$

$$(1038) \cos \frac{\theta}{2} \cdot \cos \frac{\theta}{2^2} \cdot \cos \frac{\theta}{2^3} \dots \infty$$

$$(1039) \theta \left(1 - \frac{\theta}{\pi} \right) \left(1 - \frac{\theta}{2\pi} \right) \left(1 + \frac{\theta}{\pi} \right) \left(1 - \frac{\theta}{3\pi} \right) \\ \times \left(1 - \frac{\theta}{4\pi} \right) \left(1 + \frac{\theta}{2\pi} \right) - \dots \infty$$

$$(1040) \left(1 - \frac{\sin^2 \theta}{\sin^2 \beta} \right) \left(1 - \frac{\sin^2 \theta}{\sin^2 3\beta} \right) \dots \left(1 - \frac{\sin^2 \theta}{\sin^2 (n-1)\beta} \right)$$

$$(1041) 2^{n-1} \prod_0^{n-1} \sin (\theta + ra)$$

$$(1042) \text{Lt}_{n=\infty} \left[\tan \frac{\pi}{2n} \cdot \tan \frac{2\pi}{2n} \dots \frac{\tan n\pi}{2n} \right]^{1/n}$$

Reference

$$= \frac{\sin \theta}{\theta} \quad \text{Q. 137}$$

$$= \cos \theta \quad \text{A. 216}$$

$$= \frac{2}{3}$$

$$= x! \quad \text{AD. 10}$$

$$= \sqrt{\frac{(2\pi)^{n-1}}{n}} \frac{(n \cdot x)!}{n^{nx}} \quad \text{AD. 18}$$

$$= \frac{\cosh k - \cos \theta}{1 - \cos \theta} \quad \text{Q. 137}$$

$$= \frac{\sin \theta}{\theta} \quad \text{A. 114}$$

$$= e^{-(\theta/\pi)} \log_2 \sin \theta \quad \text{Q. 35}$$

$$= \cos n\theta \quad (n \text{ even}) \quad \beta = \frac{\pi}{2n} \quad \text{A. 211}$$

$$= \sin n\theta \quad \text{where } 0 < \theta < a \quad \text{A. 211}$$

$$= 1 \quad \text{1Z. 355}$$

Series No.

$$(1043) \quad \text{Lt}_{n=\infty} \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)^{1/2} \left(1 + \frac{3}{n}\right)^{1/3} \dots \left(1 + \frac{n}{n}\right)^{1/n} \right]$$

$$(1044) \quad \frac{x+c}{c} \prod_{-\infty}^{+\infty} \left(1 + \frac{x}{n+c}\right) e^{-x/n}$$

$$(1045) \quad 2^{n-1} \prod_{r=0}^{r=n-1} \sin \left(\theta + \frac{r\pi}{n} \right)$$

$$(1046) \quad 2^{n-1} \sin \left(\theta + \frac{\pi}{2n} \right) \sin \left(\theta + \frac{3\pi}{2n} \right) \dots \sin \left(\theta + \frac{2n-1}{2n} \pi \right)$$

$$(1047) \quad 2^{n-1} \cos \theta \cos \left(\theta + \frac{\pi}{n} \right) \cos \left(\theta + \frac{2\pi}{n} \right) \dots \left(\theta + \frac{n-1}{n} \pi \right)$$

$$(1048) \quad 2^{n-1} \cos \frac{\pi}{2n} \cos \frac{3\pi}{2n} \cos \frac{5\pi}{2n} \dots \cos \frac{2n-1}{2n} \pi$$

$$(1049) \quad 2^{n-1} \sin \frac{\pi}{2n} \sin \frac{3\pi}{2n} \dots \sin \frac{2n-1}{2n} \pi$$

$$(1050) \quad \cos \frac{\pi}{n} \cos \frac{2\pi}{n} \dots \cos \left(\frac{2n-1}{n} \right) \pi$$

$$(1051) \quad \left(1 - \frac{\theta}{a}\right) \left(1 + \frac{\theta}{\pi - a}\right) \left(1 - \frac{\theta}{\pi + a}\right) \left(1 + \frac{\theta}{2\pi - a}\right) \left(1 - \frac{\theta}{2\pi + a}\right) \dots \infty$$

$$(1052) \quad \prod_0^{\infty} \left\{ 1 + \frac{\theta}{a + r\pi} \right\}$$

$$(1053) \quad \prod_0^{\infty} \left\{ 1 + \frac{2\theta}{2a + r\pi} \right\}$$

$$(1054) \quad \prod_0^{\infty} \left\{ 1 - \frac{2\theta}{2a + r\pi} \right\}$$

Reference

$$= e^{\pi^2/12} \quad 1Z. 355$$

$$= \frac{\sin \pi(x + c)}{\sin \pi c} \quad A. 224$$

$$= \sin n\theta \quad E. 145$$

$$= \cos n\theta \quad E. 145$$

$$= (-1)^{n/2} \sin n\theta \quad \text{where } n \text{ is even}$$

$$= (-1)^{(n-1)/2} \cos n\theta \quad \text{where } n \text{ is odd} \quad E. 145$$

$$= \cos \frac{n\pi}{2} \quad E. 146$$

$$= 1 \quad E. 146$$

$$= \frac{(-1)^n - 1}{2^{2n-1}} \quad E. 146$$

$$= \frac{\sin(a - \theta)}{\sin a} \quad E. 159$$

$$= \frac{\sin(a + \theta)}{\sin a} \quad \text{when } r \text{ is a positive or negative integer or zero} \\ E. 159$$

$$= \frac{\cos(a + \theta)}{\cos a} \quad \text{when } r \text{ is an odd integer, positive or negative} \\ E. 159$$

$$= \frac{\cos(a - \theta)}{\cos a} \quad \text{when } r \text{ is an odd integer, positive or negative} \\ E. 159$$

Series No.

$$(1055) \left\{1 - \left(\frac{\theta}{\pi + a}\right)^2\right\} \left\{1 - \left(\frac{\theta}{\pi - a}\right)^2\right\} \left\{1 - \left(\frac{\theta}{3\pi + a}\right)^2\right\} \\ \times \left\{1 - \left(\frac{\theta}{3\pi - a}\right)^2\right\} \dots \infty$$

$$(1056) \left(1 - \frac{\theta^2}{a^2}\right) \left\{1 - \left(\frac{\theta}{2\pi + a}\right)^2\right\} \left\{1 - \left(\frac{\theta}{2\pi - a}\right)^2\right\} \dots \infty$$

$$(1057) \left(1 - \frac{\theta}{a}\right) \left(1 - \frac{\theta}{\pi - a}\right) \left(1 + \frac{\theta}{\pi + a}\right) \\ \times \left(1 + \frac{\theta}{2\pi - a}\right) \left(1 - \frac{\theta}{2\pi + a}\right) \dots \infty$$

$$\sqrt{?} (1058) \left(\frac{3^2}{3^2 - 1}\right) \left(\frac{5^2 - 1}{5^2}\right) \left(\frac{7^2}{7^2 - 1}\right) \left(\frac{9^2 - 1}{9^2}\right) \dots \infty$$

$$(1059) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{5^2}\right) \left(1 - \frac{1}{7^2}\right) \dots \infty$$

$$(1060) \prod_1^{\infty} \frac{n^3 - x^3}{n^3 + x^3}$$

$$(1061) \prod_1^{\infty} \frac{n(n + a + b)}{(n + a)(n + b)}$$

$$(1062) 4 \cos^2 \frac{a}{2} \left\{1 + \left(\frac{\theta}{\pi + a}\right)^2\right\} \left\{1 + \left(\frac{\theta}{a - \pi}\right)^2\right\} \dots \infty$$

$$(1063) \left(1 + \frac{1}{1^2}\right) \left(1 + \frac{1}{2^2}\right) \left(1 + \frac{1}{3^2}\right) \dots \infty$$

$$(1064) \prod_{m=1}^{\infty} \left\{1 - \frac{x^2}{m^2 - a^2}\right\}$$

$$(1065) \left(1 + \frac{1}{1}\right) \left(1 - \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \infty$$

$$= \frac{\cos \theta + \cos a}{1 + \cos a} \quad \text{E. 159}$$

$$= \frac{\cos \theta - \cos a}{1 - \cos a} \quad \text{E. 159}$$

$$= \frac{\sin a - \sin \theta}{\sin a} \quad \text{E. 160}$$

$$= \frac{\{\Gamma(\frac{1}{4})\}^4}{16\pi^2} \quad \text{Q. 259}$$

$$= \frac{\pi}{4} \quad \text{A. 106}$$

$$= \frac{\Gamma(1+x)\Gamma(1+tx)\Gamma(1+t^2x)}{\Gamma(1-x)\Gamma(1-tx)\Gamma(1-t^2x)}$$

$$\text{when } t = \frac{1}{2}(-1 + i\sqrt{3})$$

$$\text{or } t^3 = 1 \quad \text{A. 313}$$

$$= \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(1+a+b)} \quad \text{A. 115}$$

$$= 2 \cosh \theta + 2 \cos a \quad \text{E. 160}$$

$$= \frac{1}{\pi} \sinh \pi \quad \text{E. 160}$$

$$= \frac{a}{\sqrt{a^2 + x^2}} \frac{\sin \{\pi \sqrt{a^2 + x^2}\}}{\sin \pi a} \quad \text{E. 161}$$

$$= 1 \quad \text{A. 108}$$

Series No.

$$(1066) \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \dots \infty$$

$$(1067) \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \dots \infty$$

$$(1068) \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \dots \infty$$

$$(1069) \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{6}\right) \dots \infty$$

$$(1070) \left(1 + \frac{1}{1 \cdot 3}\right)\left(1 + \frac{1}{3 \cdot 5}\right)$$

$$(1071) 2(1 + 2^2) \left\{1 + \left(\frac{2}{3}\right)^2\right\} \left\{1 + \left(\frac{2}{5}\right)^2\right\} \dots \infty$$

$$(1072) \left(1 + \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{3}}\right)\left(1 + \frac{1}{\sqrt{4}}\right) \dots \infty$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{3}}\right)\left(1 - \frac{1}{\sqrt{4}}\right) \dots \infty$$

$$(1073)^\dagger n \left(1 - \frac{\sin^2 \theta}{\sin^2 \alpha}\right) \left(1 - \frac{\sin^2 \theta}{\sin^2 2\alpha}\right) \dots \left\{1 - \frac{\sin^2 \theta}{\sin^2 \frac{1}{2}(n-1)\alpha}\right\}$$

$$(1074)^\dagger n \left(1 - \frac{\sin^2 \theta}{\sin^2 \alpha}\right) \left(1 - \frac{\sin^2 \theta}{\sin^2 2\alpha}\right) \dots \left\{\frac{\sin^2 \theta}{1 \sin^2 \frac{1}{2}(n-2)\alpha}\right\}$$

$$(1075)^\dagger \left(1 - \frac{\sin^2 \theta}{\sin^2 \beta}\right) \left(1 - \frac{\sin^2 \theta}{\sin^2 3\beta}\right) \dots \left\{1 - \frac{\sin^2 \theta}{\sin^2 (n-2)\beta}\right\}$$

$$(1075a)^\dagger \left(1 - \frac{\sin^2 \theta}{\sin^2 \beta}\right) \left(1 - \frac{\sin^2 \theta}{\sin^2 3\beta}\right) \dots \left\{1 - \frac{\sin^2 \theta}{\sin^2 (n-1)\beta}\right\}$$

$$(1076) 2^{n-1} \prod_0^{n-1} \left\{ \cos \theta - \cos \left(\beta + \frac{2\kappa\pi}{n} \right) \right\}$$

$$^\dagger \alpha = \frac{\pi}{n}$$

$$^\dagger \beta = \frac{\pi}{2n}$$

Reference

$$= \infty$$

A. 106

$$= 0$$

C. 159

$$= \frac{1}{2}$$

A. 106

$$= 0$$

E. 155

$$= \frac{\pi}{2}$$

A. 213

$$= e^{\pi} + e^{-\pi}$$

E. 190

$$= 0$$

A. 111

$$= \frac{\sin n\theta}{\sin \theta} \quad \text{where } n \text{ is odd}$$

A. 210

$$= \frac{\sin n\theta}{\sin \theta \cos \theta} \quad \text{where } n \text{ is even}$$

A. 210

$$= \frac{\cos n\theta}{\cos \theta} \quad \text{where } n \text{ is odd}$$

A. 211

$$= \cos n\theta \quad \text{where } n \text{ is even}$$

A. 211

$$= \cos n\theta - \cos n\beta$$

T. 84

Series No.

$$(1077) \prod_0^{n-1} \left\{ a^2 - 2ab \cos \left(\theta + \frac{2\kappa\pi}{n} \right) + b^2 \right\}$$

$$(1078) \theta \prod_1^{\infty} \left(1 + \frac{\theta^2}{n^2\pi^2} \right)$$

$$(1079) \prod_0^{\infty} \left\{ 1 + \frac{4\theta^2}{(2n+1)^2\pi^2} \right\}$$

$$(1080) \prod_1^{\infty} \cos \frac{\theta}{2^n}$$

$$(1081) \prod_0^{\infty} (1 + x^{2^n})$$

$$(1082) \prod_{-\infty}^{\infty} \left[1 - \frac{x^2}{(n+c)^2} \right]$$

$$(1083) \prod_{-\infty}^{\infty} \left[1 - \frac{4x^2}{(n\pi+x)^2} \right]$$

$$(1084) \prod_{-\infty}^{\infty} \left[e^{\theta/n\pi} \left(1 - \frac{\theta}{n\pi} \right) \right]$$

XXIV. Fourier's Series

$$(1085) \frac{f'(2\pi) - f'(0)}{n^2} - \frac{f'''(2\pi) - f'''(0)}{n^4} + \dots \infty$$

$$(1086) \frac{1}{2}f(0) + f\left(\frac{2\pi}{n}\right) + f\left(\frac{4\pi}{n}\right) + \dots + \frac{1}{2}f(2\pi)$$

$$(1087) \frac{2\pi}{n} \left\{ \frac{1}{2}f(0) + f\left(\frac{2\pi}{n}\right) + \dots + \frac{1}{2}f(2\pi) \right\} \\ - 2 \frac{f'(2\pi) - f'(0)}{n^2} \cdot \frac{\pi^2}{6} + 2 \frac{f'''(2\pi) - f'''(0)}{n^4} \cdot \frac{\pi^4}{90} \dots \infty$$

Reference

$$= a^{2n} - 2a^n b^n \cos n\theta + b^{2n} \quad \text{T. 84}$$

$$= \sinh \theta \quad \text{T. 130}$$

$$= \cosh \theta \quad \text{T. 130}$$

$$= \frac{\sin \theta}{\theta} \quad \text{T. 130}$$

$$= \frac{1}{1-x} \quad \text{where } x^2 < 1 \quad \text{T. 130}$$

$$= 1 - \frac{\sin^2 \pi x}{\sin^2 \pi c} \quad \text{A. 224}$$

$$= -\frac{\sin 3x}{\sin x} \quad \text{A. 224}$$

$$= \frac{\sin \theta}{\theta} \quad n = 0 \text{ omitted} \quad \text{A. 215}$$

$$= \int_0^{2\pi} f(t) \cos nt \, dt \quad \text{if all differential coefficients are finite and the series is convergent} \quad \text{X. 140}$$

$$= \frac{n}{2\pi} \int_0^{2\pi} f(t) \, dt + \frac{n}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt + \frac{n}{\pi} \int_0^{2\pi} f(t) \cos 2nt \, dt + \dots \infty \quad \text{X. 140}$$

$$= \int_0^{2\pi} f(t) \, dt \quad \text{where } \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty$$

$$\text{and } \frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots \infty, \text{ etc.} \quad \text{X. 140}$$

Series No.

$$\begin{aligned}
 (1088) \quad & h \left\{ \frac{1}{2} f(0) + f(h) + f(2h) + \dots + \frac{1}{2} f(nh) \right\} \\
 & - \frac{h^2}{2 \cdot 3!} \{f^1(nh) - f^1(0)\} + \frac{h^4}{6 \cdot 5!} \{f^3(nh) - f^3(0)\} \\
 & - \frac{h^6}{6 \cdot 7!} \{f^5(nh) - f^5(0)\} + \frac{3h^8}{10 \cdot 9!} \{f^7(nh) - f^7(0)\} \\
 & - \frac{5h^{10}}{6 \cdot 11!} \{f^9(nh) - f^9(0)\} + \dots \infty
 \end{aligned}$$

Series to be convergent

$$\begin{aligned}
 (1089) \quad & h \left\{ \frac{1}{2} f(x) + f(x+h) + f(x+2h) + \dots \right\} + \frac{h^2}{2 \cdot 3!} f^1(x) \\
 & - \frac{h^4}{6 \cdot 5!} f^3(x) + \frac{h^6}{6 \cdot 7!} f^5(x) - \frac{3h^8}{10 \cdot 9!} f^7(x) + \dots \infty
 \end{aligned}$$

Series to be convergent

XXV. Hypergeometric Functions

$$\begin{aligned}
 (1090) \quad & 1 + \frac{ab}{1 \cdot c} x + \frac{a(a+1)b(b+1)}{2!c(c+1)} x^2 \\
 & + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} x^3 + \dots \infty
 \end{aligned}$$

$$(1091) \quad \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$(1092) \quad F(A, 1; C, x)$$

$$\begin{aligned}
 (1093) \quad & \frac{x^{m+1}}{m+1} \left\{ 1 + \frac{m+2}{m+3} x^2 + \dots + \frac{(m+2) \dots (m+2n-2)}{(m+3) \dots (m+2n-1)} x^{2n-2} \right\} \\
 & + (1-x^2)^{-1/2} \frac{(m+2)(m+4) \dots (m+2n)}{(m+1)(m+3) \dots (m+2n-1)} \\
 & \times \int_0^x t^{m+2n} (1-t^2) - \frac{1}{2} dt
 \end{aligned}$$

$$= \int_0^{nh} f(t) dt \quad \text{X. 140}$$

$$= \int_x^\infty f(t) dt \quad \text{X. 141}$$

$$= F(a, b; c, x) \quad \text{where } |x| < 1 \quad \text{Q. 281}$$

$$= F(a, b; c, 1) \quad \text{Q. 282}$$

$$= \frac{1}{1-x} F\left(C-A, 1; C, \frac{x}{x-1}\right) \quad \text{Q. 286}$$

$$= (1-x^2)^{-1/2} \int_0^x t^m (1-t^2)^{-1/2} dt \quad \text{where } x \text{ and } 1-x \text{ are not negative real numbers} \quad \text{Q. 109}$$

*Series No.***XXVI. Relations between Products and Series**

$$(1094) (1 + xz)(1 + x^3z) \dots (1 + x^{2m-1}z)$$

$$(1095) \log h \{(1 - x)(1 - x^2)(1 - x^3) \dots \infty\}$$

$$(1096) \prod_1^{\infty} \left(1 + \frac{x}{p_n}\right)$$

$$(1097) \prod_1^m (1 + x^n z)$$

$$(1098) \prod_1^{\infty} (1 + x^n z)$$

(1099) In the following relations between series and products, all p are prime and may be related to the p_n , q_n , r_n , and t_n of series Nos. (315) to (318).

$$\text{Series (315)} \quad 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} \dots = r_n$$

$$\text{Series (316)} \quad 1 - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} \dots = t_n$$

$$\text{Series (317)} \quad 1 + \frac{1}{3^n} - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{11^n} \dots = p_n$$

$$\text{Series (318)} \quad 1 - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} \dots = q_n$$

$$= 1 + \sum_{n=1}^m \frac{(1 - x^{2m})(1 - x^{2m-2}) \dots (1 - x^{2m-2n+2}) x^{n^2} z^n}{(1 - x^2)(1 - x^4) \dots (1 - x^{2n})} \quad \text{C. 344}$$

$$= - \sum_1^{\infty} f(n) \frac{x^n}{n} \quad \text{where } |x| < 1 \text{ and } f(n) \text{ denotes the sum of all}$$

the divisors of the positive integer n ; for example,
 $f(4) = 1 + 2 + 4$ C. 345

$$= 1 + \sum_1^{\infty} \frac{x(x + p_1)(x + p_2) \dots (x + p_{n-1})}{p_1 p_2 p_3 \dots p_n} \quad \text{if the product}$$

converges to a definite limit C. 420

$$= 1 + \sum_1^m \frac{(1 - x^m)(1 - x^{m-1}) \dots (1 - x^{m-n+1})}{(1 - x)(1 - x^2) \dots (1 - x^n)} x^{\{m(n+1)\}/2} z^n$$

C. 340

$$= 1 + \sum_1^{\infty} \frac{x^{\{n(n+1)\}/2} z^n}{(1 - x)(1 - x^2) \dots (1 - x^n)} \quad \text{where } |x| < 1 \quad \text{C. 341}$$

AE. 87

$$= \frac{1}{\left(1 - \frac{1}{5^n}\right) \left(1 + \frac{1}{7^n}\right) \left(1 + \frac{1}{11^n}\right) \left(1 - \frac{1}{13^n}\right) \dots}$$

$$= \frac{1}{\left(1 + \frac{1}{5^n}\right) \left(1 + \frac{1}{7^n}\right) \left(1 - \frac{1}{11^n}\right) \left(1 - \frac{1}{13^n}\right) \dots}$$

$$= \frac{1}{\left(1 - \frac{1}{3^n}\right) \left(1 + \frac{1}{5^n}\right) \left(1 + \frac{1}{7^n}\right) \left(1 - \frac{1}{11^n}\right) \dots} \quad \text{omitting non-primes}$$

$$= \frac{1}{\left(1 + \frac{1}{3^n}\right) \left(1 + \frac{1}{5^n}\right) \left(1 - \frac{1}{7^n}\right) \left(1 + \frac{1}{11^n}\right) \dots} \quad \text{omitting non-primes}$$

Series No.

$$(1100) \prod_1^{\infty} (1 + v_n)$$

XXVII. Special Functions

$$(1101) \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{d}{dx} \log \Gamma(x)$$

$C =$ Euler's constant; see No. (1132).

$$\psi(x) = -\frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} - \dots - \frac{1}{x+n} + \psi(x+n+1)$$

$$\sum_0^{\infty} \left(\frac{1}{x+n} - \frac{1}{n+1} \right)$$

$$\sum_0^{\infty} \left(\frac{1}{y+n} - \frac{1}{x+n} \right)$$

$$\sum_1^{\infty} \left(\frac{1}{n} - \frac{1}{x+n} \right)$$

$$\sum_0^{\infty} \frac{(-1)^n}{x+n}$$

$$\log n + \sum_0^{n-1} \frac{1}{n} \psi \left(1 + \frac{r}{n} \right)$$

$$\psi(x+1) = \frac{1}{x} + \psi(x)$$

$$\psi(1-x) = \psi(x) + \pi \cot \pi x$$

$$\psi(0) = -\infty$$

Reference

$$= 1 + \sum_1^{\infty} u_n \quad \text{where } \frac{v_n}{u_n} = \frac{1}{1 + u_1 + u_2 + \dots u_{n-1}} \quad \text{T. 131}$$

and the series is convergent

$$= -C + \int_0^1 \frac{1 - t^{x-1}}{1 - t} dt \quad \text{where } x \text{ is a real and positive integer}$$

$$\left. \begin{array}{l} \text{2Z. 97-142} \\ \text{T. 132} \end{array} \right\}$$

$$= -C + \frac{1}{1} \cdot \frac{x-1}{x} + \dots \frac{1}{n} \frac{x-1}{x+n-1} \dots \infty \quad \text{2Z. 99}$$

$$= -C - \psi(x) \quad \text{T. 133}$$

$$= \psi(x) - \psi(y) \quad \text{A. 522}$$

$$= \psi(x) + C + \frac{1}{x} \quad \left. \begin{array}{l} \text{2Z. 100} \\ \text{T. 132} \end{array} \right\}$$

$$= \frac{1}{2} \left\{ \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) \right\} = \beta(x) \quad \text{T. 133}$$

$$= \int_0^1 \frac{t^x - 1}{t + 1} dt \quad \text{A. 523}$$

$$= \psi(n) \quad \text{2Z. 98}$$

$$\psi\left(\frac{1}{2}\right) = -C - 2 \log h 2$$

$$\psi\left(\frac{3}{2}\right) = 2 - C - 2 \log h 2 \quad \text{A. 522}$$

$$\psi\left(\frac{5}{2}\right) = \frac{8}{3} - C - 2 \log h 2$$

Series No.

$$\psi(1) = -C$$

$$\psi(2) = 1 - C$$

$$\psi(3) = \frac{3}{2} - C$$

$$\psi(4) = \left(1 + \frac{1}{2} + \frac{1}{3}\right) - C$$

$$\psi(n) = \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) - C$$

$$\beta(x+1) + \beta(x) = \frac{1}{x}$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}$$

$$\beta(1) = \log h 2$$

$$\beta(2) = 1 - \log h 2$$

$$\beta(3) = -\frac{1}{2} + \log h 2$$

$$\beta(4) = \frac{1}{3} + \frac{1}{2} - \log h 2$$

$$\beta\left(\frac{1}{2}\right) = \frac{\pi}{2}$$

$$\beta\left(\frac{1}{3}\right) = \log h 2 + \frac{1}{3} \pi \sqrt{3}$$

$$\beta\left(\frac{3}{2}\right) = 2 - \frac{\pi}{2}$$

$$\beta\left(\frac{2}{3}\right) = -\log h 2 + \frac{1}{3} \pi \sqrt{3}$$

$$\beta\left(\frac{5}{2}\right) = \frac{2}{3} - 2 + \frac{\pi}{2}$$

$$\text{If } \frac{d^2}{dx^2} \log h \Gamma(x) = \psi'(x)$$

Reference

2Z. 136

$$\psi\left(\frac{1}{3}\right) = -C - \frac{3}{2} \log h 3 \\ - \frac{\pi}{2\sqrt{3}}$$

$$\psi\left(\frac{2}{3}\right) = -C - \frac{3}{2} \log h 3 \\ + \frac{\pi}{2\sqrt{3}}$$

$$\psi\left(\frac{3}{4}\right) = \frac{\pi}{2} - C - 3 \log 2$$

2Z. 136

A. 522

$$\psi\left(\frac{1}{4}\right) = -\frac{\pi}{2} - C - 3 \log 2$$

T. 133

A. 523

$$= \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}$$

2Z. 98

Series No.

$$\psi'(0)$$

$$\psi'\left(\frac{1}{2}\right) = 4\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\psi'(1) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\psi'\left(\frac{3}{2}\right) = 4\left(\frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\psi'(2) = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\psi'\left(\frac{5}{2}\right) = 4\left(\frac{1}{5^2} + \frac{1}{7^2} + \dots\right)$$

$$\psi'(3) = \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

$$\psi'(\infty)$$

$$(1102) \text{ If } \frac{a!}{b!(a-b)!} = \binom{a}{b} = {}_aC_b$$

$$\sum_0^{\frac{r}{2}-p} \binom{r}{2p+2n} \binom{p+n}{n}$$

$$\sum_0^{\frac{r-1}{2}-p} \binom{r}{2p+2n+1} \binom{p+n}{n}$$

$$1 + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$$

$$1 - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n}$$

$$1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2$$

Reference
2Z. 100

$$= \infty$$

$$= \frac{\pi^2}{2}$$

$$= \frac{\pi^2}{6}$$

$$= \frac{\pi^2}{2} - 4$$

$$= \frac{\pi^2}{6} - 1$$

$$= 4 \left(\frac{\pi^2}{8} - \frac{1}{1^2} - \frac{1}{3^2} \right) = \frac{\pi^2}{2} - 4 \cdot 4$$

$$= \frac{\pi^2}{6} - \frac{5}{4}$$

$$= 0$$

$$= \frac{a(a-1)(a-2)\dots(a-b+1)}{b!}$$

T. 19

$$= \frac{r}{2(r-p)} \binom{r-p}{p} 2^{r-2p}$$

$$= \binom{r-p-1}{p} 2^{r-2p-1}$$

$$= 2^n$$

T. 19

$$= 0$$

T. 19

$$= \binom{2n}{n}$$

T. 19

Series No.

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k}$$

$$1 - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^k \binom{n}{k}$$

See also No. (189), etc.

Table of Binomial Coefficients $\binom{n}{1} = n^{c_1} = n$

$$\binom{n}{1} \binom{n}{2} \binom{n}{3} \binom{n}{4} \binom{n}{5} \binom{n}{6} \binom{n}{7} \binom{n}{8} \binom{n}{9} \binom{n}{10} \binom{n}{11} \binom{n}{12}$$

1											
2	1										
3	3	1									
4	6	4	1								
5	10	10	5	1							
6	15	20	15	6	1						
7	21	35	35	21	7	1					
8	28	56	70	56	28	8	1				
9	36	84	126	126	84	36	9	1			
10	45	120	210	252	210	120	45	10	1		
11	55	165	330	462	462	330	165	55	11	1	
12	66	220	495	792	924	792	495	220	66	12	1

XXVIII. Zeta Functions

$$(1103) \sum_{n=0}^{\infty} \frac{1}{(a+n)^s}$$

If s is a negative integer $= -m$

$$\zeta(-2m)$$

$$\zeta(1-2m)$$

$$\zeta(0)$$

Reference

$$= \binom{n+1}{k+1}$$

T. 19

$$= (-1)^k \binom{n-1}{k}$$

T. 19

T. 20

$$= \zeta(s, a)$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} dx$$

Q. 266

$$= 0$$

$$= \frac{(-1)^m B_m}{2m} \quad \text{where } m = 1, 2, 3, \text{ etc.}$$

Q. 268

$$= -\frac{1}{2}$$

Series No.

XXIX. Legendre Polynomials

(1104) If $P_0(x) + hP_1(x) + h^2P_2(x) + \dots$

where $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

and

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} x^{n-4} \dots \right]$$

then

$$\sum_{r=0}^m (-1)^r \frac{(2n-2r)!}{2^r r! (n-r)! (n-2r)!} x^{n-2r}$$

Also

$$P_n(1) = 1 \quad P_n(-1) = (-1)^n \quad P_{2n+1}(0) = 0$$

$$P_{2n}(0) = (-1)^n \frac{1 \cdot 3 \dots (2n-1)}{2 \cdot 4 \dots (2n)}$$

Finally, $P_n(x) = \omega$ is a solution of

$$P_n(x) = F\left(n+1, -n; 1; \frac{1}{2} - \frac{1}{2}x\right) = P_{-n-1}(x)$$

XXX. Special Products

(1105) If $q_0 = \prod_1^{\infty} (1 - q^{2n})$

$$q_1 = \prod_1^{\infty} (1 + q^{2n})$$

Reference

$$= (1 - 2xh + h^2)^{-1/2} \quad \text{where } |2xh - h^2| < 1 \quad \text{Q. 302}$$

$$= P_n(x) \quad \text{where } m = \frac{n}{2} \text{ or } \frac{n-1}{2} \text{ whichever is an integer}$$

$$(1 - z^2) \frac{d^2 \omega}{dx^2} - 2x \frac{d\omega}{dx} + n(n+1)\omega = 0 \quad \text{Q. 304}$$

$$= \sum_{r=0}^{\infty} \frac{(n+1)(n+2)\dots(n+r)(-n)(1-n)\dots(r-1-n)}{(r!)^2} \\ \times \left(\frac{1}{2} - \frac{1}{2}x\right)^r \quad \text{Q. 312}$$

$$\text{Then } q_0 q_3 = \prod_1^{\infty} (1 - q^n) \quad \text{where } |q| < 1 \quad \text{A. 116}$$

$$q_1 q_2 = \prod_1^{\infty} (1 + q^n) \quad \text{where } n = 1, 2, 3, \text{ etc.}$$

Series No.

$$q_2 = \prod_1^{\infty} (1 + q^{2n-1})$$

$$q_3 = \prod_1^{\infty} (1 - q^{2n-1})$$

The four products q_0 , q_1 , q_2 , and q_3 are absolutely convergent.

For example,
$$\frac{1}{(1-q)(1-q^3)(1-q^5)\dots \infty}$$

$$q_0 q_2^2 = 1 + 2q + 2q^4 + 2q^9 + \dots \infty$$

$$q_0 q_1^2 = 1 + q^2 + q^6 + q^{12} + \dots q^{n(n+1)} + \dots \infty$$

$$\frac{q_0}{q_1} = 1 - 2q^2 + 2q^8 - 2q^{18} + \dots + 2q^{2n^2} + \dots \infty$$

$$q_0 q_3 = 1 - (q + q^2) + (q^5 + q^7) - (q^{12} + q^{15}) + \dots \infty$$

$$q_2^8 = q_3^8 + 16qq_1^8$$

$$q_0^3 = 1 - 3q^2 + 5q^6 - 7q^{12} + 9q^{20} \dots$$

XXXI. General Forms

$$(1106) \sum_1^{\infty} \frac{\phi(n)x^n}{(n+a)(n+b)\dots(n+k)}$$

If the degree of $\phi(n)$ is less than that of the denominator, resolve into partial fractions and use

$$\sum_1^{\infty} \frac{x^n}{a+n} = -x^{-a} \left\{ \frac{x}{1} + \frac{x^2}{2} + \dots + \frac{x^a}{a} + \log h(1-x) \right\}$$

See also No. (370).

$$q_1 q_2 q_3 = 1$$

$$= (1 + q)(1 + q^2)(1 + q^3) \dots \infty$$

$$q_0 q_3^2 = 1 - 2q + 2q^4 - 2q^9 + \dots \infty$$

$$q_0^3 = 1 - 3q^2 + 5q^6 - 7q^{12} + 9q^{20} - \dots \infty$$

$$\frac{q_0}{q_3} = 1 + q + q^3 + q^6 + q^{10} + \dots + q^{(m(m+1))/2} + \dots \infty$$

indices being alternately of the form $\frac{1}{2}n(3n \pm 1)$

This can always be summed if convergent.

C. 246

$a, b, \dots k$ are positive or negative, unequal integers and $\phi(n)$ is an integral function of n .

If degree of $\phi(n)$ is greater than that of the denominator, it may be written

$$\psi(n)x^n + \frac{\chi(n)x^n}{(n+a)(n+b)\dots(n+k)}$$

where the degree of $\chi(n)$ is less than that of the denominator and

$$\sum \psi(n)x^n$$

may be found by partial fractions and (370)

Series No.

$$(1107) \sum_1^{\infty} \frac{\phi(n)}{(n+a)(n+b)\dots(n+k)}$$

If in No. (1106) $x = 1$, the series is not convergent unless the degree of $\phi(n)$ is less than the degree of $(n+a)\dots(n+k)$.

$$(1108) \sum_1^{\infty} (-1)^{n-1} \frac{\phi(n)}{(n+a)(n+b)\dots(n+k)}$$

If absolutely convergent,

Sum $\mathcal{S}_1$

If semi-convergent,

Sum $\mathcal{S}_2$

In the above two, the series is absolutely convergent if the degree of $\phi(n)$ is less than $(n+a)\dots(n+k)$ by two units, and semi-convergent if it is less than $(n+a)\dots(n+k)$ by one unit.

$$(1109) \sum_0^{\infty} \phi_r(n) \frac{x^n}{n!} \text{ can be summed}$$

The identity can be established

$$\phi_r(n)$$

$$\text{Then } \sum_0^{\infty} \frac{\phi_r(n)x^n}{n!}$$

Reference

C. 248

$$= - \sum_{a, b, c, \dots k} \frac{\phi(-a) \left\{ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{a} \right\}}{(b-a)(c-a) \dots (k-a)}$$

where $\sum_{a, b, c, \dots k}$ means the summation with respect to

$a, b, c, \dots k$, and $a, b, \dots k$ are positive and unequal integers, and $\phi(n)$ is an integral function of n

$a, b, \dots k$ are all positive integers and $\phi(n)$ is an integral function of n . C. 253

$$= \sum_{a, b, c, \dots k} \frac{\phi(-a) \left\{ \frac{1}{a} - \frac{1}{a-1} + \dots (-1)^{a-1} \frac{1}{1} \right\}}{(b-a)(c-a) \dots (k-a)}$$

$$= \mathfrak{S}_1 + \log 2 \sum_{a, b, c, \dots k} \frac{(-1)^a \phi(-a)}{(b-a)(c-a) \dots (k-a)}$$

if $\phi_r(n)$ is an integral function of n of the r th degree C. 234

(and see Part I, p. 107)

$$= A_0 + A_1 n + A_2 n(n-1) + \dots A_r n(n-1) \dots (n-r+1)$$

$$= A_0 \sum_0^{\infty} \frac{x^n}{n!} + A_1 x \sum_1^{\infty} \frac{x^{n-1}}{(n-1)!} + A_2 x^2 \sum_2^{\infty} \frac{x^{n-2}}{(n-2)!} + \dots$$

$$+ A_r x^r \sum_r^{\infty} \frac{x^{n-r}}{(n-r)!}$$

$$= (A_0 + A_1 x + A_2 x^2 + \dots A_r x^r) e^x$$

Series No.

To determine the constants A_0 , etc.,

divide $\phi_r(n)$ by n

$\phi_1(n)$

divide $\phi_{r-1}(n)$ by $n - 1$

$\phi_{r-1}(n)$

$\phi_1(n)$

$$(1110) \sum_0^{\infty} \phi_r(n) {}_m C_n x^n$$

The identity can be established as in (1109)

$\phi_r(n)$

Then the general term in the series is

$$\begin{aligned} \phi_r(n) {}_m C_n x^n + A_0 {}_m C_n x^n + A_1 n {}_m C_n x^n + \dots \\ + A_r n(n-1)\dots(n-r+1) {}_m C_n x^n \end{aligned}$$

Therefore

$$\begin{aligned} \sum_0^{\infty} \phi_r(n) {}_m C_n x^n &= A_0 \sum_0^{\infty} {}_m C_n x^n \\ &+ m A_1 x \sum_1^{\infty} {}_{m-1} C_{n-1} x^{n-1} + \dots \\ &+ m(m-1)\dots(m-r+1) A_r x^r \sum_r^{\infty} {}_{m-r} C_{n-r} x^{n-r} \end{aligned}$$

The constants A_0 , etc., may be evaluated as in No. (1109).

$$(1111) \frac{\sum_0^{\infty} \phi_r(n) {}_m C_n x^n}{(n+a)(n+b)\dots(n+k)}$$

This can in general be reduced to

$$\sum_0^{\infty} \frac{\psi(n) {}_{n+k} C_{m+k} x^{n+k}}{(m+1)(m+2)\dots(m+k)x^k}$$

$$\begin{aligned}
 &= \phi_{r-1}(n)n + A_0 \\
 &= \phi_{r-2}(n)(n-1) + A_1 \\
 &= A_r(n-r) + A_{r-1}
 \end{aligned}$$

where $\phi_r(n)$ is any integral function of n of the r th degree C. 195

$$= A_0 + A_1n + A_2n(n-1) + \dots + A_rn(n-1)\dots(n-r+1)$$

$$\begin{aligned}
 &\equiv A_0 {}_mC_n x^n + mA_1 x {}_{m-1}C_{n-1} x^{n-1} \\
 &\quad + m(m-1)A_2 x^2 {}_{m-2}C_{n-2} x^{n-2} + \dots \\
 &\quad + m(m-1)\dots(m-r+1)A_r x^r {}_{m-r}C_{n-r} x^{n-r}
 \end{aligned}$$

$$\begin{aligned}
 &= A_0(1+x)^m + mA_1x(1+x)^{m-1} + \dots \\
 &\quad + m(m-1)\dots(m-r+1)A_rx^r(1+x)^{m-r}
 \end{aligned}$$

$$\begin{aligned}
 &= \left\{ A_0 + \frac{mA_1x}{1+x} + \frac{m(m-1)A_2x^2}{(1+x)^2} + \dots \right. \\
 &\quad \left. + m(m-1)\dots(m-r+1) \frac{A_rx^r}{(1+x)^r} \right\} (1+x)^m
 \end{aligned}$$

where $a, b, \dots, k$ are unequal positive integers in ascending order of magnitude.

where $\psi(n)$ is an integral function of n viz. $\phi_r(n)$ multiplied by all the factors which are not absorbed by ${}_{m+k}C_{n+k}$.

C. 196

Series No.

$$(1112) \sum a^x \phi(x)$$

The series within the brackets stops at the n th difference of $\phi(x)$, supposing $\phi(x)$ to be of the n th degree.

$\phi(x)$ is rational and integral

(1113) Sum of a series whose n th term is r^n times an integral function of n of the S th degree, such as

$$\sum^n \{a_0 n^S + a_1 n^{S-1} + \dots a_r\} r^n$$

Multiply by $(1 - r)^{S+1}$. For example,

$$(1 - r)^3 \sum_1^n (1^2 r + 2^2 r^2 + \dots n^2 r^n)$$

whence by addition

$$\sum (1^2 r + 2^2 r^2 + \dots n^2 r^n)$$

(1114)† $\sum u_x$ (Approximate summation)

The constant K is to be determined in each case by substituting a known value of x .

(1115) If the sum $f(r)$ of a finite or infinite series

$f(r) = a_0 + a_1 r + a_2 r^2 + \dots$ is known, then

$$a_0 \cos x + a_1 r \cos(x + y) + a_2 r^2 \cos(x + 2y) \dots$$

$$a_0 \sin x + a_1 r \sin(x + y) + a_2 r^2 \sin(x + 2y) \dots$$

† For values of B_n^* , see No. (1129).

Reference

W. 53

$$= C + \frac{a^x}{a-1} \left\{ \phi(x) - \frac{a}{a-1} \Delta \phi(x) \right. \\ \left. + \frac{a^2}{(a-1)^2} \Delta^2 \phi(x) - \frac{a^3}{(a-1)^3} \Delta^3 \phi(x) \dots \right\}$$

where $\Delta \phi(x) = \phi(x+h) - \phi(x)$

$$= 1^2 r + 2^2 r^2 + 3^2 r^3 + \dots n^2 r^n \\ - 3r^2 - 3 \cdot 2^2 r^3 \dots \\ + 3 \cdot 1^2 r^3 + \dots \\ - 1^2 r^4 \dots - n^2 r^{n+3} \\ = \frac{r + r^2 - (n+1)^2 r^{n+1} + (2n^2 + 2n - 1)r^{n+2} - n^2 r^{n+3}}{(1-r)^3}$$

$$= K + \int u_x dx - \frac{1}{2} u_x \\ + \frac{B_1^*}{2!} \frac{du_x}{dx} - \frac{B_3^*}{4!} \frac{d^3 u_x}{dx^3} + \frac{B_5^*}{6!} \frac{d^5 u_x}{dx^5} + \dots \infty \quad 2Z. 117$$

$$= \frac{1}{2} \{ e^{ixf(re^{iy})} + e^{-ixf(re^{-iy})} \} \\ = -\frac{i}{2} \{ e^{ixf(re^{iy})} - e^{-ixf(re^{-iy})} \} \quad T. 81$$

Series No.

(1116) Euler's Summation Formula

$$\sum f(x) = f(1) + f(2) + \dots + f(x)$$

When $f(x)$ is a rational algebraic fraction or a transcendental function, this cannot necessarily be used, and the right-hand side becomes an infinite series which may not converge. (See Bromwich, Chap. XII, for a number of applications of this summation.)

(1117) If $I_a, I_a', I_a'',$ etc., are the magnitudes of the differences of the discontinuities of the function and its various differential coefficients at point a , and similarly for points b , then

$$\begin{aligned} & -\frac{1}{n} \sum I_a \sin na - \frac{1}{n^2} \sum I_a' \cos na + \frac{1}{n^3} \sum I_a'' \sin na \\ & + \frac{1}{n^4} \sum I_a''' \cos na - \dots \infty \\ & \frac{1}{n} \sum I_a \cos na - \frac{1}{n^2} \sum I_a' \sin na - \frac{1}{n^3} \sum I_a'' \cos na \\ & + \frac{1}{n^4} \sum I_a''' \sin na + \dots \infty \end{aligned}$$

XXXII. Double and Treble Series

$$(1118) \sum_0^{\infty} \sum_0^{\infty} \frac{(m+n)}{m!n!} \left(\frac{x}{2}\right)^{m+n}$$

$$(1119) \sum_0^{\infty} \sum_0^{\infty} \sum_0^{\infty} \frac{(m+n+p)!}{m!n!p!} \left(\frac{x}{3}\right)^{m+n+p}$$

Reference

$$= \int f(x) dx + \frac{1}{2}f(x) + \frac{1}{2!}B_1f'(x) - \frac{1}{4!}B_2f'''(x) + \dots \quad \text{A. 304}$$

where $f(x)$ is a polynomial and there is no term on the right-hand side (in its final form) which is not divisible by x .

$$= \pi a_n$$

(Fourier's n th harmonic amplitudes)

$$= \pi b_n \quad \text{X. 57}$$

$$\text{where } a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt \quad \text{and} \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt$$

$$\begin{aligned} \text{and } f(t) = & \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \dots \\ & + b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \dots \end{aligned}$$

$$= \frac{1}{1-x} \quad \text{where } -2 < x < 1 \quad \text{A. 194}$$

$$= \frac{1}{1-x} \quad \text{where } -3 < x < 1 \quad \text{A. 194}$$

Series No.

$$(1120) \sum_{r=2}^{\infty} \sum_{s=2}^{\infty} \frac{1}{(p+s)} r$$

$$(1121) \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{(4s-1)^{2r+1}}$$

$$(1122) \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{(4s-2)^{2r}}$$

$$(1123) \sum_{r=2}^{\infty} \sum_{s=1}^{\infty} \frac{1}{(2s)} r$$

$$(1124) \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{(4s-1)^{2r}}$$

$$(1125) \lim_{r=\infty} \left[\sum_{m=-r}^r \sum_{n=-r}^r \frac{1}{(x-m)(x-n)} \right]$$

$$(1126) \dagger \sum_{n=1}^{n=\infty} \left[\sum_{m=1}^{m=\infty} \frac{\cos 2m\pi x \cdot \cos 2n\pi y}{m^2 - n^2} \right]$$

$$(1127) \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m^2 + a^2)(n^2 + b^2)}$$

XXXIII. Bernoulli's Functions

(1128) Bernoulli Functions.

Values of $\phi_n(x)$ $\phi_n(x)$

See also No. (1129).

† See No. (1128) for values of $\phi_2(y)$.

Reference

$$= \frac{1}{p+1} \quad \text{A. 194}$$

$$= \frac{\pi}{8} - \frac{1}{2} \log 2 \quad \text{A. 194}$$

$$= \frac{\pi}{8} \quad \text{A. 194}$$

$$= \log 2 \quad \text{A. 194}$$

$$= \frac{1}{4} \log 2 \quad \text{A. 194}$$

$$= -\pi^2 \quad \text{all values } m = n \text{ are excluded} \quad \text{A. 225}$$

$$= \pi^2 \left[\frac{3}{4} \{ \phi_2(y) - \phi_2(x) \} + \frac{1}{4} \left(y - \frac{1}{2} \right) \right] \quad \text{omit } m = n \quad \text{A. 391}$$

$$= 0 \quad \text{if } x = y \quad \text{where } 0 \leq y < x \leq \frac{1}{2}$$

$$= \frac{\pi^2}{ab} \coth \pi a \coth \pi b \quad \text{Q. 136}$$

$$= x^n - \frac{n}{2} x^{n-1} + \frac{n(n-1)}{2!} B_1 x^{n-2} - \frac{n(n-1)(n-2)(n-3)}{4!} B_2 x^{n-4}$$

A. 300

terminating either in x or x^2 .

Series No.

Therefore

$$\phi_1(x) = x$$

$$\phi_2(x) = x^2 - x$$

$$\phi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

$$\phi_4(x) = x^4 - 2x^3 + x^2$$

$$\phi_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$$

$$\phi_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2$$

$$\phi_n(x+1) - \phi_n(x) = nx^{n-1}$$

$$\phi_2(x+1) = x^2 + x$$

etc.

This function is related to that in No. (1135) by $nB_n(x) = \phi_n(x)$.

(1129) Bernoulli's numbers can be calculated from the expression

$$\frac{2(2n)!}{(2\pi)^{2n}} \left\{ \frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \dots \infty \right\}$$

A similar confusion arises in the case of Euler's numbers, No. (1131). Some authorities, for the same value of n , quote the values of B_n , and others B_{2n-1} . In the tables following and in this book B_n values are used.

For example, in AC. 42, A. 297, C. 231 and AE. 3, etc.

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \text{ etc.}$$

whereas in Y. 503, etc.

$$B_1^* = \frac{1}{6}, \quad B_3^* = \frac{1}{30}, \quad B_5^* = \frac{1}{42}, \text{ etc.}$$

where $B_n = B_{2n-1}^*$.

Reference

$\phi_n(x)$ is the coefficient of $t_n/n!$ in the expansion of $t \frac{e^{xt} - 1}{e^t - 1}$ A. 300

$$= \frac{2(2n)!}{[1 - (\frac{1}{2})^{2n}](2\pi)^{2n}} \left\{ \frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \dots \infty \right\}$$

$\Rightarrow B_n = B_{2n-1}^*$

C. 231

C. 363

A. 297

Y. 503

$$\frac{2(2m)! (-1)^{m+1}}{(2^{2m} - 1) \pi^{2m}} \left[1 + \frac{1}{3^{2m}} + \frac{1}{5^{2m}} + \frac{1}{(2m+1)^{2m}} \right]$$

Extract from British Association Report, 1877 (Adams)

Table of Bernoulli's Numbers expressed in Vulgar Fractions

Numerator			$B_0 = -1$		Denominator		No.
					1	6	1
					1	30	2
					1	42	3
					1	30	4
					5	66	5
					691	2730	6
					7	6	7
					3617	510	8
					43867	798	9
				1	74611	330	10
				8	54513	138	11
				2363	64091	2730	12
				85	53103	6	13
			2	37494	61029	870	14
			861	58412	76005	14322	15
			770	93210	41217	510	16
			257	76878	58367	6	17
		26315	27155	30534	77373	1919190	18
		2	92999	39138	41559	6	19
	2	61082	71849	64491	22051	13530	20
	15	20097	64391	80708	02691	1806	21
	278	33269	57930	10242	35023	690	22
	5964	51111	59391	21632	77961	282	23
560	94033	68997	81768	62491	27547	46410	24
49	50572	05241	07964	82124	77525	66	25

Table of Bernoulli's Numbers expressed in Integers and Repeating Decimals

No.	
1	1
1	2
6	3
1	4
2	5
6	6
1	7
16	8
18	9
2	10
22	11
6	12
1	13
28	14
30	15
16	16
1	17
18	18
1	19
10	20
42	21
22	22
46	23
48	24
2	25

In the "Report of the British Association for the Advancement of Science, 1877," page 10, etc., Bernoulli's numbers are calculated up to $n = 62$. Further, the method by which they are calculated is also described.

Series No.

Values of Constants used in Series Nos. (305) to (318)

(1130)

n	0	1	2	3	4	5
B_n	-1	1/6	1/30	1/42	1/30	5/66
α_n	0	1	1	3	17	155
β_n	1	1/3	7/15	31/21	127/15	2555/33
γ_n	0	1	2	13	164	3355
δ_n	2	1	13/5	121/7	1093/5	49205/11
ϵ_n	-1	1	91/5	3751/7	1 38811/5	251 43755/11
ζ_n	0	1	13	363	18581	15 25355
η_n	0	1	14	403	20828	17 14405
θ_n	0	1	10	273	13940	11 44055
E_n	{1	1	1	2	5	16
E_n^*	{1*	1*	5*	61*	1385*	50521*
R_n	1	7	305	33367	68 15585	22374 23527
S_n	1	5	205	22265	45 44185	14916 32525
I_n	1/2	1/3	1	7	809/9	1847
H_n	3/2	3	33	903	46113	37 84503
J_n	2	10/3	34	910	4 15826/9	37 86350
T_n	0	1	23	1681	2 57543	676 37281
P_n	1	3	57	2763	2 50737	365 81523
Q_n	0	1	11	361	24611	28 73041

For the theory of Bernoulli polynomials, etc., including very many practically useful recurrence formulae, see Glaisher 1898a, c, and Lehmer 1935, 1936.

The numbers B_n , I_n , T_n , P_n , and Q_n are fundamental and are not expressible simply in terms of the others. It is to be noted that the odd Eulerian numbers are not calculable directly and are fundamental. See Edwards *Differential Calculus*, p. 502.

See No. (1131) for explanation of E_n and E_n^* .

VARIOUS NUMBERS

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Reference

AC. 42

Series No. α_n β_n γ_n δ_n ϵ_n η_n ζ_n θ_n H_n J_n E_n^* H_n I_n J_n P_n Q_n See No. (330) for q_n .† For a connection between $B_{2n+1}\left(\frac{1}{\kappa}\right)$, etc., see No. (1142).

Reference

AC. 41, 42, and 48

$$= 2(2^{2n} - 1)B_n$$

$$= (2^{2n} - 2)B_n$$

$$= \frac{3}{4}(3^{2n} - 1)B_n$$

$$= (3^{2n} - 3)B_n$$

$$= \frac{1}{2}(2^{2n} - 2)(3^{2n} - 3)B_n$$

$$= \frac{3}{8}(2^{2n} - 2)(3^{2n} - 1)B_n$$

$$= \frac{1}{3}(2^{2n} - 1)(3^{2n} - 3)B_n$$

$$= \frac{1}{4}(2^{2n} - 1)(3^{2n} - 1)B_n$$

$$= (2^{2n+1} + 1)I_n$$

$$= 2(r^{2n} + 1)I_n$$

$$= H_n + I_n$$

$$= (-1)^{n+1}4^{2n+1}B_{2n+1}\left(\frac{1}{4}\right)^{\dagger} \quad \text{AE. 51}$$

$$= (-1)^{n+1}\left\{6^{2n+1}B_{2n+1}\left(\frac{1}{6}\right) - 3^{2n+1}B_{2n+1}\left(\frac{1}{3}\right)\right\} \quad \text{AE. 51}$$

$$= (-1)^{n+1}3^{2n+1}B_{2n+1}\left(\frac{1}{3}\right) \quad \text{AE. 51}$$

$$= (-1)^{n+1}6^{2n+1}B_{2n+1}\left(\frac{1}{6}\right) \quad \text{AE. 51}$$

$$= 2^{2n}E_n^* - (2n)_2 2^{2n-2}E_{n-1}^* \dots (-1)^{n-1}(2n)_2 2^2 E_1^* + (-1)^n E_0^*$$

$$= \frac{1}{2}\left[(-1)^{n+1}8^{2n+1}B_{2n+1}\left(\frac{1}{8}\right) - E_n^*\right] \quad \text{AE. 62}$$

$$= \frac{q_n}{\sqrt{2}}(2n-1)!\left(\frac{4}{\pi}\right)^{2n} \quad \text{AE. 64}$$

Series No. R_n S_n T_n

(1131) Euler's Numbers are calculated from the equation

$$2(2n)! \left(\frac{2}{\pi}\right)^{2n+1} \left\{ \frac{1}{1^{2n+1}} - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \dots \infty \right\}$$

$$= 2(2n)! \left(\frac{2}{\pi}\right)^{2n+1} \left(1 + \frac{1}{3^{2n+1}}\right) \left(1 - \frac{1}{5^{2n+1}}\right) \left(1 + \frac{1}{7^{2n+1}}\right) \dots$$

In the list below of Euler's numbers odd values of E_n have been included for completeness. These have sometimes been called "Prepared Bernoulli Numbers," and have been omitted from many lists, and only the even numbers included. This has led to confusion because in some cases the even numbers were then called "Euler's Numbers E_n ."

In 2Z. 243 and Y. 501 Eulers' numbers are as shown in the long list below; but in AE., AC. 42, T. 141, and C. 342 and 365, the numbers are shown as E_n^* where $E_n^* = E_{2n}$

$$n = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$$

$$E_n^* = 1 \quad 1 \quad 5 \quad 61 \quad 1385 \quad 50521 \quad \text{etc.}$$

and throughout this collection E_n and E_n^* have been used to distinguish between them.

$$E_1 = 1 \quad E_2 = 1 \quad E_3 = 2 \quad E_4 = 5 \quad E_5 = 16$$

$$E_6 = 61 \quad E_7 = 272 \quad E_8 = 1385 \quad E_9 = 7936 \quad E_{10} = 50521$$

(1132) Euler's Constant

$$C = 0.57721 \ 56649 \ 01532 \ 86060 \dots$$

Reference

$$= \frac{1}{4} (3^{2n+1} + 1) E_n^* \quad \text{AC. 50}$$

$$= \frac{1}{2} (3^{2n} + 1) E_n^* \quad \text{AC. 50}$$

$$3S_n = 2R_n + E_n^*$$

$$= \frac{t_{2n}}{2\sqrt{3}} \cdot (2n-1)! \left(\frac{6}{\pi}\right)^{2n} \quad \text{AE. 75}$$

C. 342

C. 365

Y. 502

$$= E_n^* = E_{2n}$$

$$E_{12} = 270\,2765 \quad E_{14} = 1993\,60981 \quad E_{16} = 1\,93915\,12145$$

$$E_{18} = 240\,48796\,75441 \quad E_{20} = 37037\,11882\,37525$$

2Z. 245

2Z. 87

Series No.

(1133) Sum of Power Series

$$S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots \infty$$

Values of S_n to sixteen places of decimals are given in the table.

n	S_n to sixteen places of decimals
1	0.57721 56649 01532 86065 $\log \infty$ (Euler's Const. + ∞)
2	1.64493 40668 48226 4
3	1.20205 69031 59594 3
4	1.08232 32337 11138 2
5	1.03692 77551 43370 0
6	1.01734 30619 84449 1
7	1.00834 92773 81922 7
8	1.00407 73561 97944 3
9	1.00200 83928 26082 2
10	1.00099 45751 27818 0
11	1.00049 41886 04119 4
12	1.00024 60865 53308 0
13	1.00012 27133 47578 5
14	1.00006 12481 35058 7
15	1.00003 05882 36307 0
16	1.00001 52822 59408 6
17	1.00000 76371 97637 9
18	1.00000 38172 93265 0
19	1.00000 19082 12716 6
20	1.00000 09539 62033 9
21	1.00000 04769 32986 8
22	1.00000 02384 50502 7
23	1.00000 01192 19926 0
24	1.00000 00596 08189 1
25	1.00000 00298 03503 5
26	1.00000 00149 01554 8
27	1.00000 00074 50711 8

The sixteenth decimal place is not always the sixteenth occurring, but the nearest in consideration of terms to follow, e.g. C has for its 16th, 17th, etc... figures 8606....

$\log_e 10 = 2.30258 50929$
94045 6840...

Euler's Const. = $C = 0.57721$
56649 01532 8606...

$\mu = 0.43429 44819...$

SUM OF POWER SERIES

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Reference

2Z. 144

Series No.

n	S_n to sixteen places of decimals
28	1.00000 00037 25334 0
29	1.00000 00018 62659 7
30	1.00000 00009 31327 4
31	1.00000 00004 65662 9
32	1.00000 00002 32831 2
33	1.00000 00001 16415 5
34	1.00000 00000 58207 7
35	1.00000 00000 29103 8

Relations between Bernoulli's Numbers

$$(1134) \quad 1^{n-1} + 2^{n-1} + 3^{n-1} + \dots (x-1)^{n-1}$$

$$(1135) \quad \frac{x^n}{n} - \frac{1}{2}x^{n-1} + \frac{n-1}{2!}B_1x^{n-2} - \frac{(n-1)(n-2)(n-3)}{4!}B_2x^{n-4} + \dots$$

The last term is

The last term is

$$(1136) \quad B_{2n+1}(1-x)$$

$$B_{2n}(1-x)$$

$$B_n(0)$$

$$B_n(1)$$

$$(1137) \quad B_n(x) - (n-1)x B_{n-1}(x) + (n-1)x^2 B_{n-2}(x) + \dots (-1)^{n-2}x^{n-2}B_2(x) + (-1)^{n-1}x^n$$

$$(n)_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

$$(1138) \quad 1^{2n} + 2^{2n} + 3^{2n} + \dots (2x)^{2n}$$

$$1^{2n} + 3^{2n} + \dots (2x-1)^{2n}$$

$$2^{2n} + 4^{2n} + \dots + (2x)^{2n}$$

$$= B_n(x) \quad \text{where } x \text{ is a positive integer} \quad \text{AE. 8}$$

A. 304

$$= B_n(x)$$

$$(-1)^{n/2} \frac{n-1}{2} B_{n/2-1} x^2 \quad \text{where } n \text{ is even}$$

$$(-1)^{(n-1)/2} B_{(n-1)/2} x \quad \text{where } n \text{ is odd} \quad \text{AE. 7}$$

$$= -B_{2n+1}(x) \quad \text{AE. 4}$$

$$= B_{2n}(x)$$

$$= 0$$

$$= 0$$

$$= -B_n(-x) \quad \text{AE. 11}$$

$$= {}_nC_r = \left(\frac{n}{r}\right)$$

$$= B_{2n+1}(2x+1)$$

$$= B_{2n+1}(2x+1) - 2^{2n} B_{2n+1}(x+1) \quad \text{AE. 15}$$

$$= 2^{2n} B_{2n+1}(x+1)$$

Series No.(1139) $nA_n(x)$

$$A_{2n}(x)$$

$$A_n(1-x)$$

$$A_{2n+1}(0)$$

$$A_{2n}(0)$$

(1140) k is a positive integer

$$B_n(x) + B_n\left(x + \frac{1}{k}\right) + \dots + B_n\left(x + \frac{k-1}{k}\right)$$

$$(1141) \quad B_n\left(\frac{1}{k}\right) + B_n\left(\frac{2}{k}\right) + \dots + B_n\left(\frac{k-1}{k}\right)$$

$$(1142) \quad B_{2n}\left(\frac{1}{6}\right)$$

$$B_{2n}\left(\frac{1}{4}\right)$$

$$B_{2n}\left(\frac{1}{3}\right)$$

$$B_{2n}\left(\frac{1}{2}\right)$$

$$B_{2n+1}\left(\frac{1}{12}\right)$$

$$B_{2n+1}\left(\frac{1}{8}\right) - B_{2n+1}\left(\frac{3}{8}\right)$$

Reference

$= x^n - \frac{1}{2} n x^{n-1} + (n)_2 B_1 x^{n-2} - (n)_4 B_2 x^{n-4}$, etc., the series being continued so long as the exponents are not negative AE. 19

$$= B_{2n}(x) + (-1)^{n-1} \frac{B_n}{2n} \quad \text{AE. 18 and 20}$$

$$= (-1)^n A_n(x)$$

$$= A_{2n+1}(1) = 0$$

$$= A_{2n}(1) = (-1)^{n-2} \frac{B_n}{2n}$$

$$= \frac{1}{k^{n-1}} B_n(kx) \quad \text{where } n \text{ is odd} \quad \text{AE. 9}$$

$$= \frac{1}{k^{n-1}} B_n(kx) + (-1)^{n/2} \frac{k^n - 1}{k^{n-1}} \frac{B_{n/2}}{n} \quad \text{where } n \text{ is even}$$

$$= 0 \quad \text{where } n \text{ is odd} \quad \text{AE. 9}$$

$$= (-1)^{n/2} \frac{k^n - 1}{k^{n-1}} \frac{B_{n/2}}{n} \quad \text{where } n \text{ is even}$$

$$= (-1)^n \left\{ 1 + \frac{1}{2^{2n-1}} + \frac{1}{3^{2n-1}} - \frac{1}{6^{2n-1}} \right\} \frac{B_n}{4n} \quad \text{AE. 43}$$

$$= (-1)^n \left\{ 2 + \frac{1}{2^{2n-1}} - \frac{1}{4^{2n-1}} \right\} \frac{B_n}{4n} \quad \text{AE. 31}$$

$$= (-1)^n \frac{3^{2n} - 1}{3^{2n-1}} \frac{B_n}{4n} \quad \text{AE. 36}$$

$$= (-1)^n \frac{2^{2n} - 1}{2^{2n}} \frac{B_n}{n} \quad \text{AE. 26}$$

$$= \frac{1}{2} \left(1 + \frac{1}{3^{2n}} \right) B_{2n+1} \left(\frac{1}{4} \right) + \frac{1}{2^{2n+1}} B_{2n+1} \left(\frac{1}{6} \right) \quad \text{AE. 74}$$

$$= \frac{1}{2^{2n}} B_{2n+1} \left(\frac{1}{4} \right) \quad \text{AE. 63}$$

Series No.

$$B_{2n+1}\left(\frac{1}{6}\right)$$

$$6^{2n+1}B_{2n+1}\left(\frac{1}{6}\right) - 3^{2n+1}B_{2n+1}\left(\frac{1}{3}\right)$$

$$B_{2n+1}\left(\frac{1}{4}\right)$$

$$B_{2n+1}\left(\frac{1}{3}\right)$$

$$B_{2n+1}\left(\frac{1}{2}\right)$$

$$B_{2n+1}\left(\frac{3}{4}\right)$$

$$A_{2n}\left(\frac{1}{12}\right) + A_{2n}\left(\frac{5}{12}\right)$$

$$A_{2n}\left(\frac{1}{8}\right) + A_{2n}\left(\frac{3}{8}\right)$$

$$A_{2n}\left(\frac{1}{6}\right)$$

$$A_{2n}\left(\frac{1}{4}\right)$$

$$A_{2n}\left(\frac{1}{3}\right)$$

$$A_{2n}\left(\frac{1}{2}\right)$$

$$(2n + 1)4^{2n+1}B_{2n+1}\left(\frac{1}{4}\right)$$

Reference

$$= \left(1 + \frac{1}{2^{2n}}\right) B_{2n+1} \left(\frac{1}{3}\right) \quad \text{AE.52}$$

$$= (-1)^{n+1} H_n \quad \text{AE. 51}$$

$$= A_{2n+1} \left(\frac{1}{4}\right) = (-1)^{n+1} \frac{E_n^*}{4^{2n+1}} \quad \text{AE. 31}$$

$$= A_{2n+1} \left(\frac{1}{3}\right) = \frac{(-1)^{n+1}}{3^{2n+1}} I_n \quad \text{AE. 35}$$

$$= A_{2n+1} \left(\frac{1}{2}\right) = 0 \quad \text{AE. 26}$$

$$= -B_{2n+1} \left(\frac{1}{4}\right) \quad \text{AE. 63}$$

$$= \frac{1}{2^{2n-1}} A_n \left(\frac{1}{6}\right) \quad \text{AE. 77}$$

$$= \frac{1}{2^{2n-1}} A_{2n} \left(\frac{1}{4}\right) \quad \text{AE. 66}$$

$$= (-1)^n \left\{ \frac{1}{2^{2n-1}} - \frac{1}{6^{2n-1}} \right\} \frac{B_n}{4n} - A_{2n} \left(\frac{1}{3}\right) \quad \text{AE. 42}$$

$$= (-1)^n \left\{ \frac{1}{2^{2n-1}} - \frac{1}{4^{2n-1}} \right\} \frac{B_n}{4n} \quad \text{AE. 31}$$

$$= (-1)^n \left\{ 1 - \frac{1}{3^{2n-1}} \right\} \frac{B_n}{4n} \quad \text{AE. 36}$$

$$= (-1)^n \frac{2^{2n-1} - 1}{2^{2n}} \frac{B_n}{n} \quad \text{AE. 26}$$

$$= -1 + (2n+1) 2^{4^2} \left(1 - \frac{1}{2}\right) B_1 \quad \text{AE. 57}$$

$$- (2n+1) 4^{4^4} \left(1 - \frac{1}{2^3}\right) B_2$$

$$+ \dots + (-1)^{n-1} (2n+1) 4^{2^n} \left(1 - \frac{1}{2^{2^{n-1}}}\right) B_n$$

Series No.

$$(2n + 1)6^{2n+1}B_{2n+1}\left(\frac{1}{3}\right)$$

The derivation of some of these numbers is from such series as No. (576) and No. (577) by putting $\theta = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, etc.; but the original article in AE. should be consulted for a full description of the derivation.

$$(1143) \quad \frac{2}{3} \left\{ I_0 - \frac{I_1}{2!} a^2 + \frac{I_2}{4!} a^4 - \dots \infty \right\}$$

$$(1144) \quad \frac{3}{2} I_n - (2n)_2 I_{n-1} + (2n)_4 I_{n-2} + \dots \\ + (-1)^{n-1} (2n)_2 I_1 + (-1)^n I_0$$

$$(1145) \quad E_n^* - (2n)_2 E_{n-1}^* + (2n)_4 E_{n-2}^* + \dots \\ + (-1)^{n-1} (2n)_2 E_1^* + (-1)^n E_0^*$$

In No. (1142), see No. (1130) for values of I , E^* , and H .
 n_r is the binomial coefficient

$$\frac{n(n-1)\dots(n-r+1)}{r!}$$

$$(1146) \quad {}^{22}B_2\left(\frac{1}{2}\right) = -\frac{1}{2}$$

$${}^{24}B_4\left(\frac{1}{2}\right) = \frac{1}{4}$$

$${}^{26}B_6\left(\frac{1}{2}\right) = -\frac{1}{2}$$

$${}^{28}B_8\left(\frac{1}{2}\right) = \frac{51}{24}$$

Reference

$$= -1 + (2n+1)_2 6^2 \left(1 - \frac{1}{2}\right) B_1$$

AE. 51

$$- (2n+1)_4 6^4 \left(1 - \frac{1}{2^3}\right) B_2$$

$$+ \dots + (-1)^{n-1} (2n+1) 6^{2n} \left(1 - \frac{1}{2^{2n-1}}\right) B_n$$

$$= \frac{1}{1 + e^a + e^{-a}}$$

AE. 37

$$= 0$$

AE. 37

$$= 0$$

AE. 37

$$2^2 A_2 \left(\frac{1}{2}\right) = -\frac{1}{6}$$

AE. 26

$$2^4 A_4 \left(\frac{1}{2}\right) = \frac{7}{60}$$

$$2^6 A_6 \left(\frac{1}{2}\right) = -\frac{31}{126}$$

$$2^8 A_8 \left(\frac{1}{2}\right) = \frac{127}{120}$$

Series No.

$$3^2 B_2 \left(\frac{1}{3} \right) = -1$$

$$3^4 B_4 \left(\frac{1}{3} \right) = 1$$

$$3^6 B_6 \left(\frac{1}{3} \right) = -\frac{13}{3}$$

$$3^8 B_8 \left(\frac{1}{3} \right) = 41$$

$$4^2 B_2 \left(\frac{1}{4} \right) = -\frac{3}{2}$$

$$4^4 B_4 \left(\frac{1}{4} \right) = \frac{9}{4}$$

$$4^6 B_6 \left(\frac{1}{4} \right) = -\frac{33}{2}$$

$$4^8 B_8 \left(\frac{1}{4} \right) = \frac{6579}{24}$$

$$B_3 \left(\frac{1}{8} \right) = \frac{7}{8^3}$$

$$B_5 \left(\frac{1}{8} \right) = -\frac{119}{8^5}$$

Reference

AE. 36

$$3^2 A_2 \left(\frac{1}{3} \right) = -\frac{1}{4}$$

$$3^4 A_4 \left(\frac{1}{3} \right) = \frac{13}{40}$$

$$3^6 A_6 \left(\frac{1}{3} \right) = -\frac{121}{84}$$

$$3^8 A_8 \left(\frac{1}{3} \right) = \frac{1093}{80}$$

$$4^2 A_2 \left(\frac{1}{4} \right) = -\frac{1}{6}$$

AE. 31

$$4^4 A_4 \left(\frac{1}{4} \right) = \frac{7}{60}$$

$$4^6 A_6 \left(\frac{1}{4} \right) = -\frac{31}{126}$$

$$4^8 A_8 \left(\frac{1}{4} \right) = \frac{127}{120}$$

AE. 61

AE. 61

A CATALOGUE OF SELECTED DOVER BOOKS
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